Introduction

Wednesday, July 22, 2020   11:20

At the time of writing, I have just finished my notes for AP Calculus. I had made it my goal to compile a set of digital notes for Calculus and Physics C during my summer before university, for my own revision and for future AP students. I had debated which subject I should start writing notes for first, but I ultimately settled on Calculus first, as the first time I had to learn calculus for use in Physics C was a big mess and a rushed affair.

But now, with an entire year's worth of knowledge in calculus, I am about to embark on a journey to revisit Physics C, this time with all the tools I need to study it again.

Physics C was taught, at my school, within a single year, with Mechanics taking up the time before Christmas, and Electricity & Magnetism afterwards. These notes will follow the chronology as Mr Donatelli had taught the course in 2018-19, using most of his notes as the foundation of my notes. I will attempt to explain concepts in the form of equations, using as little words as possible.

Please bear with me as I revisit the most difficult AP course taught at my school, the infamous AP Physics C, and its iconic high dropout rates.

Boris Li
23 July, 2020

1 Kinematics

Thursday, July 23, 2020   15:46

Kinematics. Always the first item on every physics course's material. This unit almost always covers all material before mass and energy is introduced, and such that, this unit is limited to position, velocity, and acceleration.

1.1 The Basics

The relationship between position, velocity, and acceleration is simple. Velocity is the change in position over time, while acceleration is the change in velocity over time. Put simply:

$$v(t) = \frac{\Delta x}{\Delta t} = \frac{dx}{dt} = x'(t)$$
$$a(t) = \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = v'(t) = \frac{d^2x}{dt^2} = x''(t)$$

1.2 One-Dimensional

If we start from the acceleration equation, assuming constant acceleration:

$$a = \frac{\Delta v}{t}$$
$$at = v - v_0$$
$$v = v_0 + at$$

We reach the first provided equation on the formula sheet.

And if we develop the above equation further:

$$v = v_0 + at$$
$$\frac{dx}{dt} = v_0 + at$$
$$\int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at)dt$$
$$x - x_0 = v_0 t + \frac{1}{2}at^2$$
$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

We reach the second equation on the formula sheet, formally uniting the concepts of position, velocity, and acceleration, all in respect to time.

Now, to eliminate the variable of time:

$$v = v_0 + at$$
$$t = \frac{v - v_0}{a}$$

Inserting this expression into the second equation:

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$
$$x = x_0 + v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0^2}{a}\right)$$
$$x - x_0 = \frac{v_0 v - v_0^2}{a} + \frac{v^2 - 2v_0 v + v_0^2}{2a}$$
$$2a(x - x_0) = 2v_0 v - 2v_0^2 + v^2 - 2v_0 v + v_0^2$$
$$2a(x - x_0) = v^2 - v_0^2$$
$$v^2 = v_0^2 + 2a(x - x_0)$$

We reach the third equation that describe motion, without a time variable. This is often expressed as:

$$v^2 = v_0^2 + 2a\Delta x$$

These three equations describe one-dimensional motion, assuming constant acceleration. For higher order motion involving jerk and jounce, please derive your own equations.

1.3 Two-dimensional

When looking at two dimensional problems, split everything into two components.

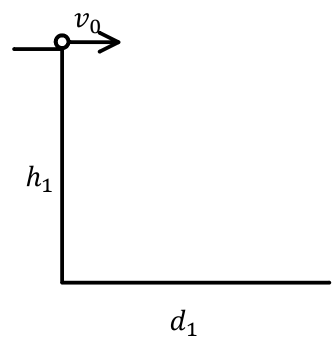
A marble travels horizontally and drops off a cliff. The cliff is h_1 tall and the marble lands d_1 away. Find the velocity as the marble leaves the top of the cliff, v_0.

Considering the horizontal components:

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$d_1 = v_0 t$$

$$t = \frac{d_1}{v_0}$$



Considering the vertical components:

$$y = y_0 + v_0 t + \frac{1}{2}at^2$$

$$h_1 = \frac{1}{2}gt^2$$

Substituting in the first expression, we get:

$$h_1 = \frac{1}{2}g\left(\frac{d_1}{v_0}\right)^2$$

$$2h_1 = \frac{gd_1^2}{v_0^2}$$

$$v_0^2 = \frac{gd_1^2}{2h_1}$$

$$v_0 = \sqrt{\frac{gd_1^2}{2h_1}}$$

When faced with a problem regarding angles, remember that sine is usually the y component, and cosine is usually the x component.

A downwards sloping ramp is constructed from the top of the cliff and hits the ground at an angle θ. A marble is dropped from the same cliff at a velocity of v_0. Determine the distance between the top of the cliff and where the marble first hits the ramp again.

If we flip the graph counterclockwise such that the ramp is horizontal, then the motion of the marble is simply projectile motion, but with gravity acting downwards and rightwards.

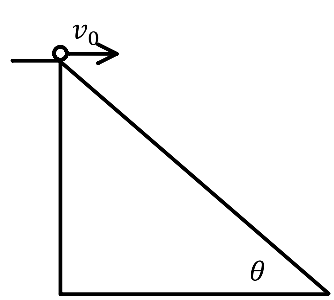
Considering the vertical components:

$$\Delta y = v_{y0}t + \frac{1}{2}a_y t^2$$

$$0 = v_0 \sin\theta\, t + \frac{1}{2}(-g\cos\theta)t^2$$

$$v_0 \sin\theta\, t = \frac{1}{2}g\cos\theta\, t^2$$

$$t = \frac{2v_0 \sin\theta}{g\cos\theta}$$



Considering the horizontal components:

$$\Delta x = v_{x0}t + \frac{1}{2}a_x t^2$$

$$\Delta x = v_0 \cos\theta\, t + \frac{1}{2}(-g \sin\theta)t^2$$

Substituting for t:

$$\Delta x = v_0 \cos\theta \left(\frac{2v_0 \sin\theta}{g \cos\theta}\right) - \frac{1}{2} g \sin\theta \left(\frac{2v_0 \sin\theta}{g \cos\theta}\right)^2$$

$$\Delta x = 2v_0^2 \sin\theta - 2v_0^2 g \frac{\sin^3\theta}{\cos^2\theta}$$

$$\Delta x = 2v_0^2 \sin\theta\,(1 - \tan^2\theta)$$

2 Laws of Motion

Sunday, July 26, 2020 10:49

2.1 Newton's Three Laws

First Law

An object stays in constant velocity or remains at rest, unless a force is acted on it.

Second Law

$\vec{F} = m\vec{a}$

Third Law

When an object exerts a force on another object, the second object exerts an equal and opposite force on the first object.

$\vec{F}_A = -\vec{F}_B$

2.2 Friction

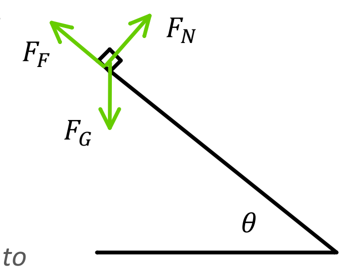
$|\vec{F}_F| = \mu|\vec{F}_N|$

An object with mass m is moving down the slope. The slope hits the ground at an angle θ, and the coefficient of friction between the slope and the object is μ. Find the acceleration of the object.

Treating the slope as the horizontal, such that the object is moving in the $+x$ direction.

Considering horizontal components:

$ma = F_G \sin\theta - F_F$
$ma = mg\sin\theta - \mu F_N$
$ma = mg\sin\theta - \mu mg\cos\theta$
$a = g(\sin\theta - \mu\cos\theta)$

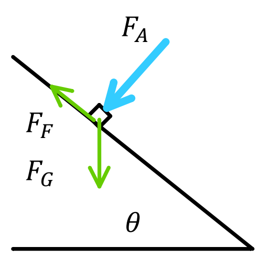


The object is now stationary on the slope a further distance down. A force F_A is applied to perpendicular to the slope to keep it stationary. Find the magnitude of F_A.

Treating the slope as the horizontal, such that the object is moving in the $+x$ direction.

Considering horizontal components:

$0 = F_G\sin\theta - F_F$
$F_G\sin\theta = F_F$
$mg\sin\theta = \mu F_N$
$mg\sin\theta = \mu(mg\cos\theta + F_A)$
$\frac{mg}{\mu}\sin\theta = mg\cos\theta + F_A$
$\frac{mg}{\mu}\sin\theta - mg\cos\theta = F_A$
$F_A = mg\left(\frac{\sin\theta}{\mu} - \cos\theta\right)$



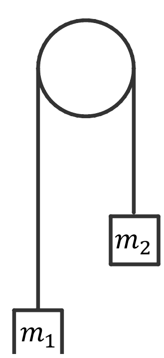
2.3 Atwood's Machines

This is assuming the pulleys are massless.

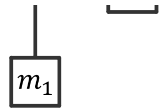
$m_1 a = m_1 g - F_T$
$F_T = m_1 g - m_1 a$

$m_2 a = F_T - m_2 g$
$F_T = m_2 g + m_2 a$

$m_1 g - m_1 a = m_2 g + m_2 a$
$g(m_1 - m_2) = a(m_1 + m_2)$
$g(m_1 - m_2)$



$$m_1 g - m_1 a = m_2 g + m_2 a$$
$$g(m_1 - m_2) = a(m_1 + m_2)$$
$$a = \frac{g(m_1 - m_2)}{m_1 + m_2}$$



2.4 Drag

Drag depends on both fluid & object properties, and is dependent on the velocity of the object. As the speed of an object increases, drag increases. And considering a falling object, if the force of gravity cancels out the force of drag, the object reaches terminal velocity.

2.4.1 Spheres

$F_D = -\alpha v - \beta v^2$

At slow speeds, $-\alpha v$ dominates.

At high speeds, $-\beta v^2$ dominates.

In general, $F_D = -k v^n$

k depends on density of fluid and shape of object.

2.4.2 Falling Objects

Assuming the falling object is dependent on v.

$$F_{net} = ma$$
$$F_G - F_D = ma$$
$$mg - \alpha v = m\frac{dv}{dt}$$
$$dt = \frac{m}{mg - \alpha v}dv$$
$$\int_0^t dt = \int_0^v \frac{m}{mg - \alpha v}dv$$
$$t = \frac{m}{-\alpha}(\ln|mg - \alpha v|)\Big|_0^v$$
$$-\frac{\alpha t}{m} = \ln|mg - \alpha v| - \ln|mg|$$
$$-\frac{\alpha t}{m} = \ln\left|\frac{mg - \alpha v}{mg}\right|$$
$$\frac{mg - \alpha v}{mg} = e^{-\frac{\alpha t}{m}}$$
$$mg - \alpha v = mge^{-\frac{\alpha t}{m}}$$
$$\alpha v = mg - mge^{-\frac{\alpha t}{m}}$$
$$v = \frac{mg}{\alpha}\left(1 - e^{-\frac{\alpha t}{m}}\right)$$

Aside:

$$\int \frac{a}{b + cx}dx$$
$$u = b + cx$$
$$du = c\,dx$$
$$= \int \frac{a}{cu}du$$
$$= \frac{a}{c}\int \frac{du}{u}$$
$$= \frac{a}{c}\ln|u| + C$$
$$= \frac{a}{c}\ln|b + cx|$$

3 Work, Energy, & Power

Tuesday, July 28, 2020 15:00

3.1 Work & Energy

Work can be described as the energy transferred when a force is applied over a distance.

$$W = \Delta E = \int \vec{F} \cdot d\vec{x}$$

3.1.0 Kinetic Energy

The derivation of the kinetic energy formula requires a form of Newton's Second Law that includes momentum, which we will discuss in the next unit.

3.1.1 Potential Function

Calculating the potential energy of an object on Earth's surface.

$$W = \int \vec{F} \cdot d\vec{x}$$

$$W_G = \int_{y_i}^{y_f} F_G \cdot dy$$

Since gravity reduces the distance between the object an Earth, and we are assuming that the positive y direction is away from the surface of the earth, gravity is negative.

$$W_G = \int_{y_i}^{y_f} (-mg) \cdot dy$$

$$W_G = -mg\Delta y$$

By definition,

$$\Delta U_G = U_f - U_i$$
$$\Delta U_G = mgy_f - mgy_i$$
$$\Delta U_G = mg\Delta y$$

Then we can conclude:

$$\therefore W_G = -\Delta U_G$$

As we set U_i to be 0, removing the reference point:

$$U_f - U_i = -\int \vec{F} \cdot d\vec{x}$$

$$U(x) \equiv U_f = -\int \vec{F} \cdot d\vec{x}$$

$$dU(x) = -F \cdot dx$$

$$F = -\frac{dU}{dx}$$

3.1.2 Newton's Law of Universal Gravitation

For any two objects, the force of gravitation should be:

$$|F_G| = \frac{Gm_1m_2}{r^2}$$

Then the work done by the force of gravitation should be:

$$W_G = \int_{x_i}^{x_f} F_G \cdot dx$$

$$\Delta U_G = -\int_{x_i}^{x_f} -\frac{Gm_1m_2}{r^2} \cdot dr$$

$$\Delta U_G = Gm_1m_2\left(-\frac{1}{r}\right)\Bigg|_{r=x_i}^{x_f}$$

$$\Delta U_G = Gm_1m_2\left(\frac{1}{x_i} - \frac{1}{x_f}\right)$$

The literal meaning of the gravitational potential energy of an object becomes the work required to bring the object from infinitely far away to a certain distance of another object.
By applying the definition of the potential function here

$$U_G = -\int_{\infty}^{r} -\frac{Gm_1m_2}{r^2} \cdot dr$$

$$U_G = Gm_1m_2\left(-\frac{1}{r}\right)\Bigg|_{\infty}^{r}$$

$$U_G = -\frac{Gm_1m_2}{r}$$

3.1.3 Hooke's Law

With this information on the potential function, we can cross-check the potential stored in a spring as according to Hooke's Law:

$$U_S = \frac{1}{2}kx^2$$

$$\frac{dU_S}{dx} = kx$$

$$F_S = -\frac{dU_S}{dx} = -kx$$

3.2 Power

Very simple. Power is defined as the amount of energy transferred over time. In other words,

$$P = \frac{dE}{dt}$$

As we know that the change is energy is just work done, we can substitute:

$$P = \frac{d(F \cdot x)}{dt}$$

And assuming force is constant over time,

$$P = F \cdot \frac{dx}{dt}$$

$$P = F \cdot v$$

4 Momentum & Centre of Mass

Wednesday, July 29, 2020 18:44

4.1 Momentum

Normally I would like to put momentum together with the previous unit, but since this is how Mr Donatelli bundled it, I am going to follow that.

Momentum is formally defined as the product of mass and velocity.

$$\vec{p} = m\vec{v}$$

A change in momentum is called impulse.

$$\vec{J} = \Delta\vec{p}$$

4.1.1 Newton's Second Law

Newton's Second Law is often presented as:

$$\vec{F} = m\vec{a}$$

But that was not how Newton had stated it. Instead, Newton had described the following, a relationship between force and momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

With a little bit of manipulation, we can alter it to become our well-known formula:

$$\vec{F} = \frac{d(m\vec{v})}{dt}$$

Assuming mass is constant over time:

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$\vec{F} = m\vec{a}$$

Then, applying the formula for impulse, we can derive:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$d\vec{p} = \vec{F} \cdot dt$$

$$\int d\vec{p} = \int \vec{F} \cdot dt$$

$$\Delta\vec{p} = \int \vec{F} \cdot dt$$

$$\vec{J} = \int \vec{F} \cdot dt$$

4.1.2 Conservation of Momentum

Newton's Second Law also implies that:

$$0 = \vec{F} = \frac{d\vec{p}}{dt}$$

If there is no external force, there will be no change in momentum. We call this conservation of momentum.

4.1.3 Kinetic Energy

$$K = \int \vec{F} \cdot d\vec{x}$$

Aside:

4.1.3 Kinetic Energy

$$K = \int \vec{F} \cdot d\vec{x}$$

$$K = \int \frac{d\vec{p}}{dt} \cdot \vec{v}dt$$

$$K = \int \vec{v} \cdot d\vec{p}$$

$$K = \int \vec{v} \cdot d(m\vec{v})$$

$$K = mv^2 - \int m\vec{v} \cdot d\vec{v}$$

Assuming mass is constant:

$$K = mv^2 - m\left(\frac{1}{2}v^2\right)$$

$$K = \frac{1}{2}mv^2$$

Aside:

$$\vec{v} = \frac{d\vec{x}}{dt}$$
$$d\vec{x} = \vec{v}dt$$

Integrating by parts:

$$\int \vec{v} \cdot d(m\vec{v})$$

$$u = \vec{v}$$
$$v = m\vec{v}$$

$$\int udv = uv - \int vdu$$

$$\int \vec{v} \cdot d(m\vec{v}) = mv^2 - \int m\vec{v} \cdot d\vec{v}$$

4.2 Centre of Mass

Normally I would put this with the next unit, since all the centre of mass calculations are about calculating the moment of inertia of objects. But again, Mr Donatelli had separated this out along with momentum, so I guess I will follow his format.

4.2.1 System of points

The centre of mass of a scattered set of points is the sum of the every product of position and mass, divided by the total mass.

$$x_{cm} = \frac{\sum x_i m_i}{m_{total}} = \frac{1}{m_{total}}(x_1m_1 + x_2m_2 + x_3m_3 + \cdots x_nm_n)$$

Calculate every coordinate separately:

$$y_{cm} = \frac{\sum y_i m_i}{m_{total}}$$
$$z_{cm} = \frac{\sum z_i m_i}{m_{total}}$$

4.2.2 Solids: Uniform Density

Use the same method as system of points.
Use the geometric centre of each uniform density object as the coordinate.
The mass densities are uniform, and are therefore constants.

$$\lambda = \frac{m}{L}\left[\frac{\text{kg}}{\text{m}}\right]$$
$$\sigma = \frac{m}{A}\left[\frac{\text{kg}}{\text{m}^2}\right]$$
$$\rho = \frac{m}{V}\left[\frac{\text{kg}}{\text{m}^3}\right]$$

4.2.3 Solids: Non-uniform density

Since masses here are continuous, we can alter the system of points formula a little bit:

$$x_{cm} = \frac{\int xdm}{m_{total}}$$

To integrate, we have to find the relationship between mass and spatial coordinates.
We express them as mass densities.

$$\lambda = \frac{dm}{dL}$$

$$\sigma = \frac{dm}{dA}$$

$$\rho = \frac{dm}{dV}$$

A rod has a mass density described by $\lambda = f(x)$. Find its centre of mass.

$$x_{cm} = \int_{x=0}^{L} \frac{xdm}{m_{total}}$$

$$x_{cm} = \frac{1}{m_{total}} \int_{x=0}^{L} xf(x)dx$$

Aside:

$$\lambda = \frac{dm}{dx}$$

$$dm = \lambda dx$$

$$dm = f(x)dx$$

5 Circular & Rotational Motion

Wednesday, July 29, 2020 20:50

5.1 Circular Motion

5.1.1 Definitions

If we travel along the circumference of a circle:

$$s = r\theta$$

Then we can obtain:

$$\frac{ds}{dt} = \frac{d(r\theta)}{dt}$$

$$v_T = r\frac{d\theta}{dt}$$

$$v_T = r\omega$$

We define:

$$\omega = \frac{d\theta}{dt}$$

And further:

$$\frac{dv_T}{dt} = r\frac{d\omega}{dt}$$

$$a_T = r\alpha$$

And define:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

5.1.2 Rotational Kinematics

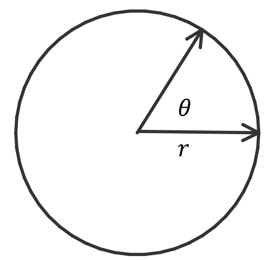
Assuming constant angular acceleration, we can apply the rotational version of the kinematic equations.

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$$

5.1.3 Centripetal acceleration



We can set the centre of the circle as the origin.
We have a position vector. In cartesian coordinates:

$$\vec{s} = r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath}$$

Since θ is a function of time, we can say:

$$\vec{v} = \frac{d\vec{s}}{dt} = -r\sin\theta\frac{d\theta}{dt}\hat{\imath} + r\cos\theta\frac{d\theta}{dt}\hat{\jmath}$$

$$\vec{v} = -r\omega\sin\theta\,\hat{\imath} + r\omega\cos\theta\,\hat{\jmath}$$

And further:

$$\vec{a} = \frac{d\vec{v}}{dt} = -r\omega\cos\theta\frac{d\theta}{dt}\hat{\imath} - r\omega\sin\theta\frac{d\theta}{dt}\hat{\jmath}$$

$$\vec{a} = -\omega^2(r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath})$$

$$\vec{a} = -\omega^2\vec{s}$$

We now know that acceleration points in the other direction of the position vector, towards the centre.
We can now express the magnitude of centripetal acceleration as:

$$a_C = \omega^2 r$$

$$a_C = \left(\frac{v^2}{r^2}\right)r$$

$$a_C = \frac{v^2}{r}$$

5.2 Rotational Kinetic Energy

From the equation for kinetic energy:

$$K = \frac{1}{2}mv^2$$

Since a rotating object has continuous mass, travelling at different speeds based on their position relative to the spinning axis:

$$K = \frac{1}{2}\sum_i m_i v_i^2$$

$$K = \frac{1}{2}\sum_i m_i (r_i\omega)^2$$

$$K = \frac{1}{2}\omega^2\sum_i m_i r_i^2$$

And we define moment of inertia, $I \equiv \sum m_i r_i^2$:

$$K = \frac{1}{2} I\omega^2$$

5.3 Moment of Inertia

Mass can be described as the magnitude of ability to resist change in translational motion.
Moment of inertia, on the other hand, describes the magnitude of ability to resist change in rotational motion.

5.3.1 Example Calculations

A massless rod with length r has two weights attached to its two ends, each weighing m. Find the moments of inertia around the centre and its ends.

$$I_{centre} = \sum m_i r_i^2$$
$$I_{centre} = m\left(\frac{r^2}{2}\right) + m\left(\frac{r^2}{2}\right)$$
$$I_{centre} = \frac{mr^2}{2}$$

$$I_{end} = \sum m_i r_i^2$$
$$I_{end} = mr^2 + 0$$
$$I_{end} = mr^2$$

A rod has length L and mass M evenly distributed throughout. Find the moments of inertia around the centre and its ends.

Around the centre:

$$I_{centre} = \int r^2 dm$$
$$I_{centre} = \int_{-L/2}^{L/2} \lambda r^2 dr$$
$$I_{centre} = \lambda \left(\frac{r^3}{3}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$
$$I_{centre} = \frac{M}{3L}\left(\frac{L^3}{8} - \left(-\frac{L^3}{8}\right)\right)$$
$$I_{centre} = \frac{ML^2}{12}$$

Aside:

$$\lambda = \frac{M}{L}$$
$$\lambda = \frac{dm}{dr}$$

At the ends:

$$I_{end} = \int r^2 dm$$
$$I_{end} = \int_0^L \lambda r^2 dr$$
$$I_{end} = \lambda \left(\frac{r^3}{3}\right)\Bigg|_0^L$$
$$I_{end} = \frac{M}{L} \cdot \frac{L^3}{3}$$
$$I_{end} = \frac{ML^2}{3}$$

A hollow cylinder has a radius of R and a mass of M, all gathered at the curved surface, distributed evenly. Find the moment of inertia as the cylinder spins arounds the centre of the open ends.

$$I = \int r^2 dm$$
$$I = R^2 \int dm$$
$$I = MR^2$$

A cylinder with mass M evenly distributed is hollowed out at the centre. The hollowed part has a radius of R_1, and the overall radius is R_2. Find the moment of inertia as the cylinder spins around the centre of the open ends.

$$I = \int r^2 dm$$

$$I = \int_{R_1}^{R_2} r^2 \,(2\pi r h\rho \cdot dr)$$

$$I = 2\pi h\rho \int_{R_1}^{R_2} r^3 dr$$

$$I = 2\pi h \frac{M}{V}\left(\frac{r^4}{4}\right)\Bigg|_{R_1}^{R_2}$$

$$I = \frac{\pi h M}{2(\pi R_2^2 h - \pi R_1^2 h)}(R_2^4 - R_1^4)$$

$$I = \frac{M}{2}\cdot\frac{(R_2^4 - R_1^4)}{(R_2^2 - R_1^2)}$$

$$I = \frac{M}{2}\cdot\frac{(R_2^2 - R_1^2)(R_2^2 + R_1^2)}{(R_2^2 - R_1^2)}$$

$$I = \frac{M}{2}(R_2^2 + R_1^2)$$

Aside:

$$\rho = \frac{M}{V}$$

$$\rho = \frac{dm}{dV}$$

$$\rho = \frac{dm}{2\pi r h \cdot dr}$$

A solid cylinder has mass M and radius R. Find the moment of inertia as it rotates around its centre, along its height.

$$I = \int r^2 dm$$

$$I = \int_{0}^{R} r^2 \,(2\pi r h\rho \cdot dr)$$

$$I = 2\pi h\rho \int_{0}^{R} r^3 dr$$

$$I = 2\pi h \frac{M}{V}\left(\frac{r^4}{4}\right)\Bigg|_{0}^{R}$$

$$I = \frac{1}{2}\pi h \frac{M}{\pi R^2 h} R^4$$

$$I = \frac{MR^2}{2}$$

5.3.2 Parallel Axis Theorem

A simpler way to find moments of inertia even when the object is not rotating around its centre of mass. For a second, let us imagine the equation in cartesian coordinates:

$$I = \int r^2 dm$$

$$I_{centre} = \int (x^2 + y^2) dm$$

Pythagorean theorem:
$r^2 = x^2 + y^2$

Imagine the new axis of rotation is x and y from the centre:

$$I = \int \left((x + x')^2 + (y + y')^2\right) dm$$

$$I = \int x^2\, dm + \int 2xx' dm + \int {x'}^2 dm + \int y^2\, dm + \int 2yy' dm + \int {y'}^2 dm$$

Since the distances are constants:

$$I = x^2 \int dm + 2x \int x' dm + \int {x'}^2 dm + y^2 \int dm + 2y \int y' dm + \int {y'}^2 dm$$

And reorganizing the terms:

$$I = \left(x^2 \int dm + y^2 \int dm\right) + \left(\int ({x'}^2 + {y'}^2) dm\right) + \left(2x \int x' dm + 2y \int y' dm\right)$$

$$I = D^2 M + I_{centre} + 0$$

$\therefore I = I_{CM} + D^2 M$

5.4 Torque

Torque is defined as the cross product of radius and force.
Cross product means perpendicular.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Because the centripetal force acts on the same direction as the radius, it does not produce torque.
However, when a tangential force is applied:

$$\Sigma F_T = ma_T$$
$$\Sigma F_T r = ma_T r$$
$$\Sigma \tau = m(r\alpha)r$$
$$\Sigma \tau = mr^2\alpha$$
$$\Sigma \tau = I\alpha$$

Similarly, for continuous objects:

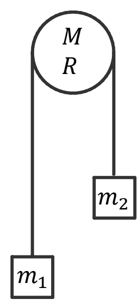
$$dF_T = a_T dm$$
$$rdF_T = ra_T dm$$
$$d\tau = r^2 \alpha dm$$
$$\int d\tau = \alpha \int r^2 dm$$
$$\tau_{net} = I\alpha$$

5.4.1 Atwood's Machines

Same thing, but when the pulleys have mass, you have to take into account their moments of inertia.

$$m_1 a = m_1 g - T_1$$
$$T_1 = m_1 g - m_1 a$$

$$m_2 a = T_2 - m_2 g$$
$$T_2 = m_2 g + m_2 a$$



$$\tau_{net} = I\alpha$$
$$R(T_1 - T_2) = \frac{MR^2}{2} \cdot \frac{a}{R}$$
$$m_1 g - m_1 a - m_2 g - m_2 a - \frac{Ma}{2} = 0$$
$$m_1 a + m_2 a + \frac{Ma}{2} = m_1 g - m_2 g$$
$$a = \frac{m_1 g - m_2 g}{m_1 + m_2 + \frac{M}{2}}$$

5.4.1 Rolling and Slipping

When a ball is rolling, translational motion is equal to its rotational counterpart:

$$x = r\theta$$
$$v = r\omega$$
$$a = r\alpha$$

When a ball is slipping, it can fall into two categories:

1. The ball is spinning, but it is not moving (as much)
   $$x < r\theta$$
   $$v < r\omega$$
2. The ball is moving, but not spinning (as much)
   $$x > r\theta$$
   $$v > r\omega$$

A ball with mass M and radius R has an initial speed of v_0 and no rotational velocity. If the coefficient of friction is μ, find the time needed for the ball to go from sliding to rolling.

Considering translational motion:

$$F_{net} = ma$$
$$-F_F = Ma$$
$$-\mu M g = Ma$$
$$a = -\mu g$$

$$v = v_0 + at$$
$$v = v_0 - \mu g t$$

Considering rotational motion:

$$\tau_{net} = I\alpha$$
$$RF_F = \frac{2}{5} MR^2 \alpha$$
$$\mu M g = \frac{2}{5} MR\alpha$$
$$\alpha = \frac{5\mu g}{2R}$$

$$\mu Mg = \frac{2}{5}MR\alpha$$
$$\alpha = \frac{5\mu g}{2R}$$

$$\omega = \omega_0 + \alpha t$$
$$\omega = \frac{5\mu g}{2R}t$$

$$v = v_0 + at$$
$$v = v_0 - \mu g t$$

When rolling:
$$v = r\omega$$
$$v_0 - \mu g t = R\frac{5\mu g}{2R}t$$
$$v_0 = \mu g t\left(1 + \frac{5}{2}\right)$$
$$t = \frac{2v_0}{7\mu g}$$

5.5 Angular Momentum

Angular momentum is defined as:
$$\vec{L} = \vec{r} \times \vec{p}$$
It is simply the rotational equivalent of momentum.
And since:
$$L = \Sigma r_i m_i v_i$$
$$L = \Sigma r_i m_i (r_i \omega)$$
$$L = \omega \Sigma (m_i r_i^2)$$
$$L = I\omega$$
And also:
$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\vec{\tau} = \vec{r} \times \frac{d\vec{p}}{dt}$$
$$\vec{\tau} = \frac{d(\vec{r} \times \vec{p})}{dt}$$
$$\vec{\tau} = \frac{d\vec{L}}{dt}$$
From the equation above, we know that, like linear momentum, angular momentum is conserved.

A rod with mass m_R and length r is fixed at one end, like a pendulum. The rod is initially at rest, and a ball with mass m_B, travelling at v_0 then collides with the non-fixed end of the rod at a right angle. The ball sticks to the end of the rod after the collision. Find the speed of the ball after the collision.

Linear momentum is not conserved here, since after the collision, the axis of rotation applies a force on the rod.
On the other hand, angular momentum is conserved here, since the force that the axis of rotation applies on the rod acts at a distance of zero, and therefore does not contribute to torque.
$$\vec{\tau} = \frac{d\vec{L}}{dt} = 0$$
$$L_i = L_f$$
$$r m_B v_0 = L_B + L_R$$
$$r m_B v_0 = r m_B v_f + I\omega$$
$$r m_B v_0 = r m_B v_f + \frac{1}{3} m_R r^2 \frac{v_f}{r}$$
$$m_B v_0 = v_f\left(m_B + \frac{m_R}{3}\right)$$
$$v_f = \frac{m_B v_0}{m_B + \frac{m_R}{3}}$$

6 Oscillations & Gravity

Sunday, August 2, 2020 18:37

6.1 Kepler's Laws

Johannes Kepler was a 17th century German dude who came up with laws about planetary motion when studying data about the motion of Mars around the Sun.
Like Newton's Laws, there are three:

1. Planetary orbits are ellipses. The sun will always be on one of them.
2. Connect the sun and the planet with a line. That line sweeps out equal areas in equal amounts of time.
3. $T^2 \propto r^3$

6.1.1 Kepler's Second Law

A planet-sun system, in isolation, normally does not have external forces acting upon it. Therefore, angular momentum is conserved.
That means, as the planet gets closer to the sun, its speed increases, and vice versa.

$$\vec{L} = \vec{r} \times m\vec{v}$$
$$\vec{r} \propto \frac{1}{\vec{v}}$$

Then it also makes sense for when the planet travels slower and farther from the sun, should cover the same area as compared to when the planet travels faster and closer to the sun.

6.1.2 Kepler's Third Law

Assuming circular motion:

$$F_{net} = ma$$
$$\frac{GmM}{r^2} = m\frac{v^2}{r}$$
$$v = \sqrt{\frac{GM}{r}}$$

And period, the time needed to complete a full revolution:

$$T = \frac{2\pi}{\omega}$$
$$T = \frac{2\pi r}{v}$$
$$T = \frac{2\pi r}{\sqrt{\frac{GM}{r}}}$$
$$T = 2\pi r\sqrt{\frac{r}{GM}}$$
$$T^2 = \frac{2\pi}{GM}r^3$$

This concept can be extended to ellipses by replacing radius with the semi-major axis.

6.2 Gravity

6.2.1 Gravitational Potential Energy

We have already discussed this topic in 3.1.2.

$$\Delta U_G = Gm_1m_2\left(\frac{1}{x_i} - \frac{1}{x_f}\right)$$

$$U_G = -\frac{Gm_1 m_2}{r}$$

6.2.2 Escape Velocity

Assuming no atmospheric drag:

$$U_i + K_i = U_f + K_f$$

$$-\frac{Gmm_\oplus}{r_\oplus} + \frac{1}{2}mv_0^2 = -\frac{Gmm_\oplus}{r_\oplus + h} + \frac{1}{2}mv_f^2$$

To completely escape the gravitational field, we will need infinite separation.
And to obtain the minimum velocity needed, the final velocity will be 0.

$$-\frac{Gmm_\oplus}{r_\oplus} + \frac{1}{2}mv_0^2 = -\lim_{h\to\infty}\frac{Gmm_\oplus}{r_\oplus + h}$$

$$\frac{1}{2}mv_0^2 = \frac{Gmm_\oplus}{r_\oplus} - \lim_{h\to\infty}\frac{Gmm_\oplus}{r_\oplus + h}$$

$$\frac{1}{2}v_0^2 = \frac{Gm_\oplus}{r_\oplus}$$

$$v_0 = \sqrt{\frac{2Gm_\oplus}{r_\oplus}}$$

6.3 Simple Harmonic Motion

6.3.1 The Second-degree Differential Equation

Simple harmonic motion, by definition, is when the restoring force is proportional to the displacement:

$$ma = -Cr$$

$$m\frac{d^2r}{dt^2} = -Cr$$

Does this look familiar?
Hooke's law states that:

$$ma = -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

We can solve for the differential equation:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

And the general solution to that is:

$$x(t) = A\cos(\omega t + \varphi)$$

Where $\omega^2 = \frac{k}{m}$

6.3.2 Small Angle Approximation

Before we move onto examples, we must first discuss this.
For this segment, O denotes opposite, A denotes adjacent, H denotes hypotenuse, and S denotes arc length with adjacent as radius.
For a very small angle:

As $\theta \to 0, O \to 0$

$$H \approx A$$

$$\sin\theta = \frac{O}{H} \approx \frac{O}{A} = \tan\theta$$

$$\tan\theta = \frac{O}{A} \approx \frac{S}{A} = \frac{\theta A}{A} = \theta$$

$$\therefore \sin\theta \approx \theta$$

Or a simpler explanation:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$
$$\sin 0 = 0$$
$$\therefore \sin \theta \approx \theta$$

6.3.3 Simple Pendulum

To solve for a motion of a simple pendulum:

$$F = ma$$
$$-mg \sin \theta = m \frac{d^2 s}{dt^2}$$

Using small angle approximation:

$$-g\theta = \frac{d^2(\theta r)}{dt^2}$$
$$-\frac{g}{r}\theta = \frac{d^2\theta}{dt^2}$$

And using the general solution:

$$\theta(t) = A \cos(\omega t + \varphi)$$

Where $\omega^2 = \frac{g}{r}$

6.3.4 Torsional Pendulum

If there are linear pendulums, there must be torsional pendulums.
To solve for their motion:

$$\tau = I\alpha$$
$$-k\theta = I \frac{d^2\theta}{dt^2}$$
$$-\frac{k}{I}\theta = \frac{d^2\theta}{dt^2}$$

And using the general solution:

$$\theta(t) = A \cos(\omega t + \varphi)$$

Where $\omega^2 = \frac{k}{I}$

Conclusion

Thursday, August 6, 2020 15:16

Many have enrolled themselves in the course the year before. Many have regretted that decision and dropped out before school started. Many have been deterred by the extreme pace that this course requires, and have switched over to Physics 2 or Physics 12, in search of easier material. Many have felt dejected when they first see a 30% on their first test.

A few will have triumphed. A few will have persevered. A few will have suffered. Maybe you are one of them. Maybe you are not. But if you have reached here, you must be tired after all those physics concepts and differential equations. Take a break. Do not be afraid to stop. Physics isn't for everyone.

If you are still reading, congratulations. Mechanics was a massive hurdle, but from here onwards, it is only an uphill battle. Brace yourself for more strenuous times, and I wish you good luck.

Boris Li
6 August, 2020

Appendix: Moments of Inertia

Sunday, August 2, 2020 17:26

Sphere

Solid, rotating around centre

Moment of inertia of a disk:

$$I = \frac{1}{2}MR^2$$

Treating a sphere as infinitely many disks stacked together:

$$dI = \frac{1}{2}r^2 dm$$

$$dI = \frac{1}{2}\rho\pi r^4 dx$$

$$dI = \frac{1}{2}\rho\pi(R^2 - x^2)^2 dx$$

$$I = \int_{-R}^{R} \frac{1}{2}\rho\pi(R^2 - x^2)^2 dx$$

$$I = \frac{1}{2}\pi\frac{M}{V}\int_{-R}^{R}(R^4 - 2R^2x^2 + x^4)dx$$

$$I = \frac{1}{2}\pi\frac{M}{V}\left(R^4x - \frac{2}{3}R^2x^3 + \frac{x^5}{5}\right)\Bigg|_{-R}^{R}$$

$$I = \frac{1}{2}\pi\frac{M}{V}\left(2R^5 - \frac{4}{3}R^5 + \frac{2}{5}R^5\right)$$

$$I = \frac{1}{2}\pi\frac{3M}{4\pi R^3}\left(\frac{16}{15}R^5\right)$$

$$I = \frac{2}{5}MR^2$$

Aside:

$$\rho = \frac{M}{V}$$

$$\rho = \frac{dm}{dV}$$

x being the distance on the axis, from the centre:

$$dm = \rho(\pi r^2 dx)$$

x, r, and R readily form a triangle:

$$R^2 = x^2 + r^2$$

$$r^2 = R^2 - x^2$$