Introduction

Thursday, July 23, 2020 00:25

I often think that Mechanics burns you out, and E&M deals the killing blow in Physics C. Without a proper foundation in AP Physics 1, and a firm grasp on most basic concepts, Mechanics can often introduce unnecessary variables (ahem, calculus) and complicate your journey through physics. I would say I have fallen into this trap, not doing any revision the summer between Physics 1 and Physics C.

On the other hand, the material taught in Electricity & Magnetism are not purely E&M, but also involve mechanics concepts, such as tension and density. Unlike mechanics, it is often difficult to visualize concepts in E&M, and if you have stumbled along the path of Mechanics, navigating through E&M can only be described as full of tripwires and obstacles.

I will attempt to convey concepts as clearly as possible, as to help future physics students. If you are ever lost, I hope this document can ever guide you so slightly.

Boris Li
6 August, 2020

7 Electrostatics

Thursday, August 6, 2020 15:25

Electrostatics. Stuff that doesn't move.
In the following sections, we define:

Coulomb's constant:

$$k_E = \frac{1}{4\pi\varepsilon_0}$$

Magnetic constant:

$$k_M = \frac{\mu_0}{4\pi}$$

Where ε_0, vacuum permittivity, and μ_0, vacuum permeability, are physical constants that are related as such:

$$\varepsilon_0\mu_0 = \frac{1}{c^2}$$

7.1 Coulomb's Law

Coulomb's Law is basically Newton's Law of Universal Gravitation but for point charges.
Force is proportional to the inverse square of distance, giving us the equation:

$$\left|\vec{F}_E\right| = \left|\frac{k_E q_1 q_2}{r^2}\right|$$

Two insulating balls with mass m and charge q are separated by a distance r, and are hanging from the ceiling by two pieces of string of negligible mass. The balls are in equilibrium and are not in motion.

Considering horizontal components:

$$F_T \cos\theta = F_E$$

$$F_T \cos\theta = \frac{k_E q^2}{r^2}$$

Considering vertical components:

$$F_T \sin\theta = F_G$$

$$F_T \sin\theta = mg$$

7.2 Electric Field

A gravitational field vector can be defined as:

$$F_G = \frac{Gm_1m_2}{r^2} = \vec{g}_1(r)m_2$$

Where $\vec{g}_1(r) = \frac{Gm_1}{r^2}$

g_1 is the gravitational field created by m_1.

Same can be said for an electric field:

$$F_E = \frac{k_E q_1 q_2}{r^2} = \vec{E}_1(r)q_2$$

Where $\vec{E}_1(r) = \frac{k_E q_1}{r^2}$

E_1 is the electric field created by q_1.

7.2.1 Electric Field Lines

Few rules:

1. Goes from positive to negative charges

2. Higher density of lines indicate a larger electric field, and a larger charge.

7.2.2 Continuous Charges

A rod of length L lies on the x-axis has charge $+Q$ evenly distributed throughout. Find the electric field at a point a away from the end of the rod on the x-axis.

$$dE = \frac{k_E dq}{r^2}$$

$$E = k_E \int \frac{dq}{r^2}$$

$$E = k_E \lambda \int_a^{a+L} \frac{dr}{r^2}$$

$$E = \frac{Q}{4\pi\varepsilon_0 L}\left(-\frac{1}{a+L} + \frac{1}{a}\right)$$

$$E = \frac{Q}{4\pi\varepsilon_0 L}\left(\frac{1}{a} - \frac{1}{a+L}\right)$$

Aside:

$$\lambda = \frac{Q}{L}$$

$$\lambda = \frac{dq}{dr}$$

A rod of infinite length that lies on the x-axis has charge density of $\lambda = \frac{\lambda_0 x_0}{x}$. Find the electric field at a point x_0 away from the end of the rod on the x-axis.

$$dE = \frac{k_E dq}{r^2}$$

$$E = k_E \int \frac{dq}{r^2}$$

$$E = k_E \int_{x_0}^{\infty} \frac{\lambda_0 x_0}{x^3} dx$$

$$E = k_E \lambda_0 x_0 \left(\lim_{b\to\infty}\left(-\frac{1}{2b^2}\right) - \left(-\frac{1}{2x_0^2}\right)\right)$$

$$E = \frac{\lambda_0 x_0}{8\pi\varepsilon_0 x_0^2}$$

Aside:

$$\lambda = \frac{dq}{dr}$$

$$\frac{\lambda_0 x_0}{x} dx = dq$$

A ring of charges with radius a centered around the origin lies on the yz-plane with a total charge of $+Q$. Find the electric field at a point x away from the origin on the x-axis.

We know that the y and z components cancel each other out.

Finding the x-component:

$$dE = \frac{k_E dq}{r^2}$$

$$dE_x = \frac{k_E dq}{x^2 + a^2} \cos\theta$$

$$dE_x = \frac{k_E dq}{x^2 + a^2} \cdot \frac{x}{\sqrt{x^2 + a^2}}$$

$$E = \int \frac{k_E x dq}{(x^2 + a^2)^{\frac{3}{2}}}$$

$$E = \frac{x}{4\pi\varepsilon_0 (x^2 + a^2)^{\frac{3}{2}}} \int dq$$

$$E = \frac{Qx}{4\pi\varepsilon_0 (x^2 + a^2)^{\frac{3}{2}}}$$

Pythagorean theorem:

$$r^2 = x^2 + a^2$$

Aside:

$$\lambda = \frac{Q}{s} = \frac{Q}{2\pi a}$$

$$\lambda = \frac{dq}{ds}$$

$$dq = \lambda ds$$

$$dq = \lambda a d\theta$$

$$\int dq = \lambda a \int_0^{2\pi} d\theta$$

$$\int dq = \lambda a 2\pi$$

$$\int dq = \frac{Q}{2\pi a} 2\pi a$$

$$\int dq = \lambda a 2\pi$$

$$\int dq = \frac{Q}{2\pi a} 2\pi a$$

$$\int dq = Q$$

A disk of charges with radius r centered around the origin lies on the yz-plane with a total charge of $+Q$. Find the electric field at a point x away from the origin on the x-axis.

Treating the disk as an infinite number of rings:

$$dE = \frac{x}{4\pi\varepsilon_0 (x^2 + r^2)^{\frac{3}{2}}} dq$$

$$dE = \frac{x}{4\pi\varepsilon_0 (x^2 + r^2)^{\frac{3}{2}}} 2\pi r \sigma dr$$

$$E = \frac{2\pi\sigma x}{4\pi\varepsilon_0} \int_0^r \frac{r}{(x^2 + r^2)^{\frac{3}{2}}} dr$$

$$E = \frac{\sigma x}{2\varepsilon_0} \int_{x^2}^{x^2+r^2} \frac{1}{2} u^{-\frac{3}{2}} du$$

$$E = \frac{\sigma x}{2\varepsilon_0} \left(-\frac{1}{\sqrt{u}}\right)\Bigg|_{x^2}^{x^2+r^2}$$

$$E = \frac{\sigma x}{2\varepsilon_0} \left(\frac{1}{x} - \frac{1}{\sqrt{x^2 + r^2}}\right)$$

$$E = \frac{Q}{2\pi r^2 \varepsilon_0} \left(\frac{x}{x} - \frac{x}{\sqrt{x^2 + r^2}}\right)$$

$$E = \frac{Q}{2\pi r^2 \varepsilon_0} \left(1 - \frac{x}{\sqrt{x^2 + r^2}}\right)$$

Aside:

$$\sigma = \frac{Q}{A} = \frac{Q}{\pi r^2}$$

$$\sigma = \frac{dq}{dA}$$

$$\sigma = \frac{dq}{2\pi r dr}$$

u/du substitution:

$$u = x^2 + r^2$$

$$du = 2rdr$$

A cylinder of charges with radius r and height $x_2 - x_1$ centered around the x-axis, has both ends lying on the yz-plane with a total charge of $+Q$. Find the electric field at a point x_1 away from the closer end of the cylinder.

Treating the cylinder as an infinite stack of disks:

$$dE = \frac{1}{2\pi r^2 \varepsilon_0} \left(1 - \frac{x}{\sqrt{x^2 + r^2}}\right) dq$$

$$dE = \frac{\pi r^2 \rho}{2\pi r^2 \varepsilon_0} \left(1 - \frac{x}{\sqrt{x^2 + r^2}}\right) dx$$

$$E = \frac{\rho}{2\varepsilon_0} \int_{x_1}^{x_2} \left(1 - \frac{x}{\sqrt{x^2 + r^2}}\right) dx$$

$$E = \frac{\rho}{2\varepsilon_0} \left(\int_{x_1}^{x_2} dx - \int_{x_1^2+r^2}^{x_2^2+r^2} \frac{1}{2} u^{-\frac{1}{2}} du\right)$$

$$E = \frac{\rho}{2\varepsilon_0} \left((x_2 - x_1) - (\sqrt{u})\Big|_{x_1^2+r^2}^{x_2^2+r^2}\right)$$

$$E = \frac{Q}{2\pi r^2 (x_2 - x_1)\varepsilon_0} \left(x_2 - x_1 - \sqrt{x_2^2 + r^2} + \sqrt{x_1^2 + r^2}\right)$$

Aside:

$$\rho = \frac{Q}{V} = \frac{Q}{\pi r^2 (x_2 - x_1)}$$

$$\rho = \frac{dq}{\pi r^2 dx}$$

u/du substitution:

$$u = x^2 + r^2$$

$$du = 2xdx$$

7.3 Electric Flux

7.3.1 Definition

Flux can be described as the amount of rays that pass through a certain area.
Flux is determined by two variables:

1. Area, in which by increasing it, increases flux;
2. The amount of rays, i.e. the electric field, in which by increasing it, more rays are produced, and flux is increased.

It can be described in this equation:

$$\Phi_E = \vec{E} \cdot \vec{A}$$

Beware that flux only considers the rays that pass through, so as the area tilts, the perpendicular area that rays can pass through decreases, and flux is diminished.
When the area is completely parallel to the direction of the electric field, no flux is present.

For a closed surface, the only way to obtain positive flux is to put the object inside the closed surface, since there will always be a net amount of electric field lines leaving the enclosed space.
It is impossible to obtain negative flux, since putting an object outside of the enclosed surface will result in every single electric field line entering and leaving the closed surface, resulting in a net electric flux of zero.

7.3.2 Gauss' Law

If there is an equation that describes stuff algebraically, there must be a calculus version.

$$\Phi_E = \vec{E} \cdot \vec{A}$$

$$d\Phi_E = \vec{E} \cdot d\vec{A}$$

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A}$$

A point charge is at the centre of a sphere with radius r. Find the electric flux through the sphere.
Using the definition of electric flux:

$$\Phi_E = \vec{E} \cdot \vec{A}$$

The electric field lines are always perpendicular to the surface of the sphere.

$$\Phi_E = \vec{E}\vec{A}$$

$$\Phi_E = \frac{k_E q}{r^2} 4\pi r^2$$

$$\Phi_E = \frac{q}{4\pi\varepsilon_0} 4\pi$$

$$\Phi_E = \frac{q}{\varepsilon_0}$$

Gauss' Law describes exactly this relationship.

$$\Phi_E = \oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

Gauss' Law can be used to find the electric field, once we choose a Gaussian surface enclosing that charge. Tips:

1. $\vec{E} \cdot d\vec{A}$ should equal 0 ($\vec{E} \perp d\vec{A}$) or $\vec{E} d\vec{A}$ ($\vec{E} \parallel d\vec{A}$).
2. For $\vec{E} \parallel d\vec{A}$ scenarios, $\vec{E}$ should be constant.

For an infinitely long rod of charges with uniform charge density of λ, find the electric field at a point r away from the rod.
Using a cylinder as our Gaussian surface:

$$\int \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

For the ends of the cylinder, no electric field is present.
For the sides of the cylinder, the electric field is perpendicular to the surface.

$$\vec{E}\int d\vec{A} = \frac{\lambda L}{\varepsilon_0}$$

$$2\pi r L\vec{E} = \frac{\lambda L}{\varepsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\pi r\varepsilon_0}$$

Aside:

$$\lambda = \frac{Q}{L}$$

For an infinite plane of charges with uniform charge density of σ, find the electric field at a point r away from the plane.

Using a cube as our Gaussian surface:

$$\int \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

For the sides of the cube, no electric field is present.
For the top and bottom of the cube, the electric field is perpendicular to the surface.

$$\vec{E}\int d\vec{A} = \frac{\sigma A}{\varepsilon_0}$$

$$2\vec{E}\vec{A} = \frac{\sigma A}{\varepsilon_0}$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}$$

Aside:

$$\sigma = \frac{Q}{A}$$

The electric field is constant everywhere.

For an insulating sphere of radius R with uniform charge density of ρ, find the electric field both inside and outside of the sphere. Express in terms of r, the distance from the centre of the sphere.

Using a sphere as our Gaussian surface:
Inside the sphere:

$$\int \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

The electric field is perpendicular to every part of the spherical surface.

$$\vec{E}\int d\vec{A} = \frac{4\pi r^3\rho}{3\varepsilon_0}$$

$$4\pi r^2\vec{E} = \frac{4\pi r^3\rho}{3\varepsilon_0}$$

$$\vec{E} = \frac{r\rho}{3\varepsilon_0}$$

Aside:

$$\rho = \frac{Q}{V}$$

$$\rho = \frac{Q}{\frac{4}{3}\pi r^3}$$

Outside the sphere:

$$\int \vec{E}\cdot d\vec{A} = \frac{q}{\varepsilon_0}$$

The electric field is perpendicular to every part of the spherical surface.

$$\vec{E}\int d\vec{A} = \frac{4\pi R^3\rho}{3\varepsilon_0}$$

$$4\pi r^2\vec{E} = \frac{4\pi R^3\rho}{3\varepsilon_0}$$

$$\vec{E} = \frac{R^3\rho}{3r^2\varepsilon_0}$$

Aside:

$$\rho = \frac{Q}{V}$$

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3}$$

The combined answer can be expressed as:

$$\vec{E} = \begin{cases} \dfrac{r\rho}{3\varepsilon_0}, r < R \\ \dfrac{R^3\rho}{3r^2\varepsilon_0}, r \geq R \end{cases}$$

8 Conductors, Capacitors, & Dielectrics

Monday, August 10, 2020 17:13

8.1 Electric Potential

8.1.1 Comparison

Considering gravitational potential energy:
The energy needed to move from point a to point b is:

$$\Delta U_G = -W_G$$

$$\Delta U_G = -\int \vec{F}_G \cdot d\vec{r}$$

$$\Delta U_G = -\int_{r_A}^{r_B} -\frac{GMm}{r^2} \cdot dr$$

$$\Delta U_G = -\frac{GMm}{r}\bigg|_{r_A}^{r_B}$$

$$\Delta U_G = GMm\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

We can say this is the change in gravitational potential energy when moving an object m in regards to another object M.
However, if we take away mass m and only look at object M, we can say:

$$\Delta V_G = GM\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

This is the change in gravitational potential, i.e. the change in energy per unit mass.

Considering electric potential energy:
The energy needed to move from point a to point b is:

$$\Delta U_E = -W_E$$

$$\Delta U_E = -\int \vec{F}_E \cdot d\vec{r}$$

$$\Delta U_E = -\int_{r_A}^{r_B} \frac{k_E Qq}{r^2} \cdot dr$$

$$\Delta U_E = \frac{k_E Qq}{r}\bigg|_{r_A}^{r_B}$$

$$\Delta U_E = k_E Qq\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

We can say this is the change in electric potential energy when moving an object q in regards to another object Q.
However, if we take away charge q and only look at object Q, we can say:

$$\Delta V_E = k_E Q\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$

This is the change in electric potential, i.e. the change in energy per unit charge.

8.1.2 Definition

Change in electric potential is defined as:

$$\Delta V = \frac{\Delta U_E}{q}$$

And because … blah blah blah … definition … blah blah … reciprocal … blah blah … infinity … zero:

$$\Delta V = k_E Q \left(\frac{1}{r_B} - \lim_{r_A \to \infty} \frac{1}{r_A} \right)$$

$$\Delta V = k_E Q \frac{1}{r_B}$$

$$V = \frac{k_E Q}{r}$$

And multiple distinct charges:

$$V = \sum \frac{k_E Q}{r}$$

And continuous charges:

$$V = \int \frac{k_E dq}{r}$$

Simple, right?

Given a semi-ring of charge with radius R and total charge of Q, find the electric potential at the centre.

$$V = \int \frac{k_E dq}{r}$$

$$V = \int \frac{k_E \lambda dL}{R}$$

$$V = \frac{k_E \lambda}{R} \int dL$$

$$V = \frac{k_E Q}{\pi R^2} \pi R$$

$$V = \frac{Q}{4\pi R \varepsilon_0}$$

Aside;

$$\lambda = \frac{Q}{L} = \frac{Q}{\pi R}$$

$$\lambda = \frac{dq}{dL}$$

8.1.3 Equipotential Lines

Lines that show the same electric potential around an object.
Recall that to traverse a volt means a joule of energy needed to move a coulomb of charge.
Electric field lines are perpendicular to equipotential lines.
Tighter equipotential lines mean stronger electric field.

$$\Delta V = \frac{\Delta U}{q}$$

$$\Delta V = \frac{-W}{q}$$

$$\Delta V = \frac{-\int \vec{F} \cdot d\vec{r}}{q}$$

$$\Delta V = \frac{-\int q\vec{E} \cdot d\vec{r}}{q}$$

$$\Delta V = -\int \vec{E} \cdot d\vec{r}$$

$$dV = -\vec{E} \cdot d\vec{r}$$

$$\vec{E} = -\frac{dV}{d\vec{r}}$$

i.e. An larger increase in electric potential over the same distance results in a stronger electric field, and vice versa.

So, given an equipotential chart, we can approximate:

$$\vec{E} \approx -\frac{\Delta V}{\Delta \vec{r}}$$

8.1.4 Conductors

Free electrons can move freely inside conductors. As a result, it will always be in a state of electrostatic equilibrium. Excess charge will be forced to the surface.

1. Electric field inside a conductor is 0.
2. Electric field lines will always be perpendicular to the surface of the conductor.
3. Electric field outside of the conductor can be treated as if the conductor is a point charge at its centre.

This implies:

1. Electric potential inside the conductor is constant.
2. Electric potential at the surface is also the same constant.

8.2 Capacitance

8.2.1 Mechanism

A capacitor stores charge via two conductors separated by a insulating region.
The capacitance of a capacitor can be defined as:

$$C = \left|\frac{Q}{V}\right|$$

Where one plate has a charge of $+Q$ and the other $-Q$.

Note that the capacitance of a capacitor stays constant even as the voltage and the amount of charge changes.

8.2.2 Electric Fields

An electric field is produced by the two separated parallel plates:

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

Aside:

$$\sigma = \frac{Q}{A}$$

For the positive plate:
Using a cube as our Gaussian surface:

$$E \cdot 2A = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0}$$

We know that, in a circuit, the negative plate should have the same charge, and since parallel plates have the same area, they have the same electric field.

$$E_{total} = \frac{\sigma}{\varepsilon_0}$$

Therefore, the change in electric potential is:

$$\Delta V = -\int E \cdot dr$$

$$\Delta V = -E \int dr$$

$$\Delta V = -Ed$$

$$\Delta V = -\frac{\sigma d}{\varepsilon_0}$$

$$\Delta V = -\frac{Qd}{\varepsilon_0 A}$$
$$-\frac{Q}{\Delta V} = \frac{\varepsilon_0 A}{d}$$

We can say, by altering the area of the plate and the distance between the plates, capacitance can be altered:

$$C = \frac{\varepsilon_0 A}{d}$$

8.2.3 Dielectrics

Dielectrics align themselves in the presence of an electric field.
Since opposites attract, negative charges are attracted to the positive plate, and positive charges are attracted to the negative plate.
Since electric fields flow from positive to negative charges, an electric field in the opposite direction is produced, and the net electric field strength is reduced.
And from this equation, we know:

$$\Delta V = -\int E \cdot dr$$

When electric field strength decreases, the difference is electric potential also decreases.
But in the presence of a battery, the potential difference between the plates must match the voltage of the battery, and such that, the electric field produced by the plates must increase in strength to match the battery.
And from this equation, we know:

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

When the electric field of the plates increase, and the area of the plates remain unchanged, charge must increase.

$$C = \left|\frac{Q}{V}\right|$$

And with charge increasing, electric potential staying constant, capacitance must increase.
To take into account the effect of a dielectric, we alter the equation:

$$C = \frac{\kappa \varepsilon_0 A}{d}$$

Where κ is the dielectric constant, the multiplicative factor that a dielectric material has on capacitance.

8.2.4 Energy Stored

The electrical energy stored by a capacitor can be expressed as the work done on the capacitor:
Work here means work done by the capacitor:

$$W = -\Delta U$$
$$W = -q\Delta V$$

Work here means work done on the capacitor, to eliminate the negative sign:

$$dW = -\Delta V dq$$
$$dW = -\frac{q}{C} dq$$
$$\int_0^W dW = \int_0^Q -\frac{q}{C} dq$$
$$W = U_C = \frac{Q^2}{2C} = \frac{1}{2} Q\Delta V = \frac{1}{2} C\Delta V^2$$

9 Circuits

Tuesday, August 11, 2020 15:05

9.1 Current & Resistance

9.1.1 Current

Current as defined as the amount of charge that passes per unit time:

$$I = \frac{dQ}{dt}$$

If we look at a wire of length Δx and cross-sectional area A:

$$\Delta Q = \text{number of charges} \cdot \text{elementary charge}$$
$$\Delta Q = \text{volume} \cdot \frac{\text{number of charges}}{\text{volume}} \cdot \text{elementary charge}$$
$$\Delta Q = A\Delta x \cdot N \cdot e$$
$$I\Delta t = Av_d \Delta t N e$$
$$I = Nev_d A$$

9.1.2 Ohm's Law

Current density is defined as:

$$J = \frac{I}{A}$$

Aside:

$$\rho = \frac{1}{\sigma}$$

And Ohm's Law states that:

$$J = \sigma E$$
$$E = \rho J$$

To obtain the common form of Ohm's Law:

$$\Delta V = -\int E \cdot dr$$
$$\Delta V = E\ell$$
$$\Delta V = \frac{J}{\sigma}\ell$$
$$\Delta V = \frac{I\ell}{\sigma A}$$
$$\Delta V = I\left(\frac{\ell}{\sigma A}\right)$$
$$\Delta V = IR$$

Where resistance is defined as:

$$R = \frac{\ell}{\sigma A} = \frac{\rho \ell}{A}$$

9.1.3 Power

Power is defined as:

$$P = \frac{dU}{dt}$$
$$P = \frac{d}{dt}(q\Delta V)$$
$$P = \frac{dQ}{dt}\Delta V + \frac{dV}{dt}Q$$
$$P = I\Delta V + 0$$
$$P = I\Delta V$$

Alternate forms:

$$P = I^2 R = \frac{(\Delta V)^2}{R}$$

9.2 Kirchoff's Rules

9.2.1 Loop Rule & Junction Rule

The loop rule states that:

The sum of voltages in a closed loop is 0.

The junction rule states that:

The sum of currents entering and leaving a junction is 0.

9.2.2 Resistors

For resistors connected in a parallel circuit:

$$I_T = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} + \cdots$$

$$R_P = \left(\frac{I}{\Delta V_T}\right)^{-1} = \left(\sum \frac{1}{R}\right)^{-1}$$

For resistors connected in a series circuit:

$$\Delta V_T = IR_1 + IR_2 + IR_3 + \cdots$$

$$R_S = \frac{\Delta V_T}{I} = \sum R$$

9.2.3 Capacitors

For capacitors connected in a parallel circuit:

$$Q = C\Delta V$$

$$\frac{dQ}{dt} = C\frac{dV}{dt}$$

$$I_T = C_1\frac{dV}{dt} + C_2\frac{dV}{dt} + C_3\frac{dV}{dt} + \cdots$$

$$C_P = \frac{Q}{\Delta V} = \int \frac{Idt}{dV} = \sum C$$

For capacitors connected in a series circuit:

$$\Delta V_T = \frac{Q_T}{C_1} + \frac{Q_T}{C_2} + \frac{Q_T}{C_3} + \cdots$$

$$C_S = \left(\frac{\Delta V}{Q}\right)^{-1} + \left(\sum \frac{1}{C}\right)$$

9.3 Charging

9.3.1 Electromotive Force

Electromotive force is the maximum voltage that a battery can output.

For a real battery, there must be an internal resistance, which is defined as the following:

$$\Delta V = \varepsilon - Ir$$

The voltage represented here is the actual voltage output.

9.3.2 Capacitor Charging

Given a circuit consisting of a battery ε, a resistor R, a capacitor C, and a switch. The switch is initially open, and the capacitor is uncharged. Find the charge on the positive plate and the current through the circuit as a function of time as the switch is closed.

$$\varepsilon - IR - V_C = 0$$

$$\varepsilon - \frac{dq}{dt}R - \frac{q}{C} = 0$$
$$\frac{dq}{dt}R = \varepsilon - \frac{q}{C}$$
$$\frac{dq}{\varepsilon - \frac{q}{C}} = \frac{dt}{R}$$

u/du substitution:
$$u = \varepsilon - \frac{q}{C}$$
$$du = -\frac{1}{C}dq$$

$$\frac{-Cdu}{u} = \frac{dt}{R}$$
$$-C\int_{q=0}^{q}\frac{du}{u} = \frac{1}{R}\int_{0}^{t}dt$$
$$-C\ln u\Big|_{\varepsilon}^{\varepsilon-\frac{q}{C}} = \frac{t}{R}$$
$$\ln\frac{\varepsilon - \frac{q}{C}}{\varepsilon} = -\frac{t}{RC}$$
$$\frac{\varepsilon - \frac{q}{C}}{\varepsilon} = e^{-\frac{t}{RC}}$$
$$1 - \frac{q}{C\varepsilon} = e^{-\frac{t}{RC}}$$
$$\frac{q}{C\varepsilon} = 1 - e^{-\frac{t}{RC}}$$
$$q = C\varepsilon\left(1 - e^{-\frac{t}{RC}}\right)$$
$$I = \frac{dq}{dt} = \frac{C\varepsilon}{RC}e^{-\frac{t}{RC}}$$
$$I = \frac{\varepsilon}{R}e^{-\frac{t}{RC}}$$

The rate of charging is determined by the exponent term.
When $t = RC$, the exponent equals to -1.
We define the time constant as:
$$\tau = RC$$

9.3.3 Capacitor Discharging

The battery is removed from the circuit, and the capacitor is now charged. A switch, initially open, replaces the battery. Find the charge on the positive plate and the current through the circuit as a function of time as the switch is closed.

$$-IR - V_C = 0$$
$$-\frac{dq}{dt}R - \frac{q}{C} = 0$$
$$\frac{dq}{dt}R = -\frac{q}{C}$$
$$\frac{dq}{q} = \frac{dt}{-RC}$$
$$\int_{Q_{max}}^{q}\frac{dq}{q} = -\frac{1}{RC}\int_{0}^{t}dt$$
$$\ln\frac{q}{Q_{max}} = -\frac{t}{RC}$$
$$q = Q_{max}e^{-\frac{t}{RC}}$$

Since a fully charged capacitor is when time approaches infinity in the previous section, we can determine that $q = C\varepsilon$:
$$q = C\varepsilon e^{-\frac{t}{RC}}$$

$$I = -\frac{\varepsilon}{R} e^{-\frac{t}{RC}}$$

10 Magnetic Fields

Thursday, August 13, 2020 18:58

In Physics C, when we refer to the magnetic field, we exclusively refer to the B-field, and not its counterpart, H-field, in of which the M-field is composed of.

10.1 Magnetic Force

10.1.1 Force on a Charged Particle

When a charged particle moves through a magnetic field, a magnetic force is applied on the particle. The force is:

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

In which the three vectors point in the 3 directions perpendicular to each other.
The directions can be summarized with the right-hand rule, assuming the direction of conventional current:

1. The force points upwards, with your thumb;
2. The velocity points forward, with your index; and
3. The magnetic field points leftward, with your middle.

As long as your fingers remain perpendicular to each other, rotate your entire hand to find the 3 directions.
When asked about the direction of electron flow instead, use your left hand.

10.1.2 Force on a Wire

We can extend the concept of a charged particle into a continuous flow of charged particles in a wire:

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

The total force must equal the forces on each charge multiplied by the number of charges.

$$\vec{F}_B = q(\vec{v} \times \vec{B})nAL$$
$$\vec{F}_B = n(qvA\vec{L}) \times \vec{B}$$
$$\vec{F}_B = I\vec{L} \times \vec{B}$$

Aside:
$$I = Nev_dA$$
$$I = nqv_dA$$

And if the wire is not always straight:

$$\vec{F}_B = I\int d\vec{L} \times \vec{B}$$

The right hand rule still applies here:

1. The force points upwards, with your thumb;
2. The length/current points forward, with your index; and
3. The magnetic field points leftward, with your middle.

A wire sits on the xy-plane, and runs along the circumference of a semicircle of radius R, before completing the loop by running across the diameter on the x-axis. A magnetic field of strength B travels from -y towards +y. Find the force the magnetic field exerts on the wire loop if a current of I travels counterclockwise in the wire loop.
Splitting the wire into two segments, the diameter and the circumference.
Considering the diameter:

$$F_1 = ILB$$
$$F_1 = I(2R)B$$
$$F_1 = 2IRB$$

From the right hand rule, the force points out of the page in the +z direction.
Considering the semicircle:

$$F_2 = I \int dL \times B$$

Since we are only taking the perpendicular portions:

$$F_2 = I \int dL \sin\theta \, B$$

$$F_2 = IB \int_0^{\pi} R \sin\theta \, d\theta$$

$$F_2 = IBR(-\cos\pi + \cos 0)$$

$$F_2 = 2IRB$$

From the right-hand rule, since the perpendicular portion of the current points left, the force must point into the page in the -z direction.

$$\vec{F}_B = F_1 + F_2 = 0$$

10.2 Magnetic Field

While we have talked about the existence of a magnetic field, we haven't discussed where they came from.

10.2.1 Biot-Savart Law

In early 19th century, there was a dude named Oersted who was playing with apparatus demonstrating electric currents. There happened to be a compass beside that apparatus. He noticed how the compass pointed in a different direction when current was generated.
Later, dudes named Biot and Savart performed more experiments determining the exact strength of magnetic fields produced by these currents. The result is the following equation:

$$d\vec{B} = k_M I \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

$$d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

The right-hand rule still applies:

1. The magnetic field points upwards, with your thumb;
2. The length/current of the wire points forward, with your index; and
3. The direction of from the wire to the location points leftward, with your middle.

A loop of wire following the circumference of a circle with radius a lies on the yz-plane, with the centre lying on the origin. When viewed from the right side, looking towards the -x direction, a current I travels counterclockwise. Find the magnetic field strength a distance x away from the the origin on the x-axis.

$$d\vec{B} = k_M I \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

$$d\vec{B} = k_M I \frac{d\vec{\ell}}{a^2 + x^2}$$

Since we know that the circle applies a magnetic field in every angle, the yz-component is canceled out.
Considering we take the the x-component of the B-field, and the current travels perpendicular to the radius, the $\hat{r}$-component must be parallel to the radius:

$$\vec{B} = k_M I \int \frac{d\vec{\ell}}{a^2 + x^2} \cos\theta$$

$$\vec{B} = k_M I \int \frac{d\vec{\ell}}{a^2 + x^2} \frac{a}{\sqrt{a^2 + x^2}}$$

$$\vec{B} = k_M I \frac{a}{(a^2+x^2)^{\frac{3}{2}}} \int_0^{2\pi a} d\vec{\ell}$$
$$\vec{B} = \frac{\mu_0}{4\pi} I \frac{a}{(a^2+x^2)^{\frac{3}{2}}} 2\pi a$$
$$\vec{B} = \frac{\mu_0 I a^2}{2(a^2+x^2)^{\frac{3}{2}}}$$

10.2.2 Ampère's Law

That was really complicated right? Well, just as how Gauss' Law simplified the method to find electric fields, Ampère's Law simplified the method to find magnetic fields.

Consider a wire of infinite length with current I. Find the magnetic field at a distance a away from the wire.

$$d\vec{B} = k_M I \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

By the right-hand rule, since the $\hat{r}$-component is parallel to the radius, and the magnetic field must be always perpendicular to the radius and the wire, and therefore run in circles around the wire.

$$\vec{B} = k_M I \int_{-\infty}^{+\infty} \frac{d\vec{\ell}}{\ell^2+a^2} \frac{a}{\sqrt{\ell^2+a^2}}$$

u/du substitution:
$\ell = a \tan u$
$d\ell = a \sec^2 u \, du$

$$\vec{B} = k_M I a 2 \int_0^{\infty} \frac{d\vec{\ell}}{(\ell^2+a^2)^{\frac{3}{2}}}$$
$$\vec{B} = 2k_M I a \int_0^{\frac{\pi}{2}} \frac{a \sec^2 u \, du}{(a^2 \tan^2 u + a^2)^{\frac{3}{2}}}$$
$$\vec{B} = 2k_M I a \int \frac{a \sec^2 u \, du}{a^3 (\tan^2 u + 1)^{\frac{3}{2}}}$$
$$\vec{B} = 2k_M I a \int \frac{a \sec^2 u \, du}{a^3 \sec^3 u}$$
$$\vec{B} = 2k_M I a \int \frac{du}{a^2 \sec u}$$
$$\vec{B} = \frac{2k_M I}{a} \int \cos u \, du$$
$$\vec{B} = \frac{2k_M I}{a} \left(\sin\frac{\pi}{2} - \sin 0 \right)$$
$$\vec{B} = \frac{\mu_0 I}{2\pi a}$$

Let us derive Ampère's Law.

For $\vec{B} \cdot d\vec{s}$, $\vec{s}$ being the circular path around the wire, $\vec{B} \parallel d\vec{s}$, therefore:

$$\vec{B} \cdot d\vec{s} = B ds$$

We also know along a circular path, with the same radius, $\vec{B}$ is constant:

$$\oint \vec{B} \cdot d\vec{s}$$

$$= \oint B ds$$

$$= B \oint ds$$

$$= \frac{\mu_0 I}{2\pi a} \oint ds$$

$$= \frac{\mu_0 I}{2\pi a} 2\pi a$$

$$= \mu_0 I$$

For an Amperian Loop (analogous to Gaussian surface),

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I$$

Now that we have the knowledge of the magnetic field outside of the wire, let us find the magnetic field inside the wire.

Consider a wire of infinite length, radius R, with current I. Find the magnetic field at a distance a away from the centre of the wire, where $a < R$.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$\vec{B}(2\pi a) = \mu_0 \frac{\pi a^2}{\pi R^2} I$$

$$\vec{B} = \frac{\mu_0 I a}{2\pi R^2}$$

Be aware that when choosing an Amperian loop, magnetic field should be constant to simplify calculations.

10.2.3 Solenoids

Notice if a current runs through a solenoid, the magnetic field outside of the solenoid is approximately zero.

Choosing a circular path around the outside and the centre of the solenoid:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{enc}$$

$$\vec{B} \int d\vec{s} = \mu_0 n L I$$

$$\vec{B} L = \mu_0 n I$$

Where L is the length of the solenoid, and n is the number of loops per unit length.

11 Electromagnetism

Saturday, August 15, 2020 14:44

11.1 Electromagnetic Induction

11.1.1 Magnetic Flux

Just as how electric flux is defined as the electric field going though an area, magnetic flux is defined as:

$$\Phi_B = \oiint \vec{B} \cdot d\vec{A}$$

11.1.2 Faraday's Law & Lenz's Law

If a current can induce a magnetic field, then a magnetic field must be able to induce a current.
Faraday's Law states that:

$$|\varepsilon| = \left|\frac{d\Phi_B}{dt}\right|$$

A change in magnetic flux will create an electromotive force.
And Lenz's Law states that the current produced opposes the change in magnetic flux.
In other words:

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

In which with the right-hand rule:

1. The magnetic field points upwards, with your thumb; and
2. The current goes clockwise, opposite to your fingers curling counterclockwise.

11.2 Inductors

11.2.1 Inductance

An inductor is a coil of wire that can store energy in a magnetic field given a current, and can generate a current when the energy is released.
Inductance is defined as the ratio between those quantities:

$$L = \frac{\Phi_B}{I}$$

Plugging this equation into Faraday's law:

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

$$\varepsilon = -\frac{d(LI)}{dt}$$

$$\varepsilon = -L\frac{dI}{dt}$$

We have now formulated a relationship between inductance, voltage, and the changing current.

11.2.2 Inductor Charging

Given a circuit consisting of a battery ε, a resistor R, an inductor L, and a switch. The switch is initially open, and the inductor is uncharged. Find the current through the circuit as a function of time as the switch is closed.

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

$$L\frac{dI}{dt} = \varepsilon - IR$$

$$\int^{I} \quad dI \qquad \int^{t} dt$$

u/du substitution:

$$L\frac{dI}{dt} = \varepsilon - IR$$

$$\int_0^I \frac{dI}{\varepsilon - IR} = \int_0^t \frac{dt}{L}$$

u/du substitution:
$u = \varepsilon - IR$
$du = -RdI$

$$\int_{\varepsilon}^{\varepsilon - IR} \frac{du}{-Ru} = \frac{t}{L}$$

$$\ln(\varepsilon - IR) - \ln\varepsilon = -\frac{Rt}{L}$$

$$\frac{\varepsilon - IR}{\varepsilon} = e^{-\frac{Rt}{L}}$$

$$\varepsilon - IR = \varepsilon e^{-\frac{Rt}{L}}$$

$$IR = \varepsilon\left(1 - e^{-\frac{Rt}{L}}\right)$$

$$I = \frac{\varepsilon}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

11.2.3 Inductor Discharging

The same inductor is now charged. The battery is replaced by a open switch. As the switch is closed, find the current through the circuit as a function of time.

$$-IR - L\frac{dI}{dt} = 0$$

$$L\frac{dI}{dt} = -IR$$

$$\int_{I_0}^{I} \frac{dI}{I} = -\frac{R}{L}\int_0^t dt$$

$$\ln\frac{I}{I_0} = -\frac{Rt}{L}$$

$$\frac{I}{I_0} = e^{-\frac{Rt}{L}}$$

$$I = I_0 e^{-\frac{Rt}{L}}$$

Where $I_0 = \frac{\varepsilon}{R}$.

11.2.4 Energy Stored

Since we know that voltages:

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$

And we can deduce that power $P = I\Delta V$:

$$I\varepsilon - I^2R - IL\frac{dI}{dt} = 0$$

The power of the inductor is:

$$P = IL\frac{dI}{dt}$$

And since:

$$I = \frac{dQ}{dt}$$

$$I\Delta V = \frac{d(QV)}{dt}$$

$$P = \frac{dU}{dt}$$

The energy stored by an inductor is:

$$\frac{dU}{dt} = IL\frac{dI}{dt}$$
$$dU = ILdI$$
$$\int dU = \int_0^I ILdI$$
$$\Delta U = \frac{1}{2}LI^2$$

11.2.5 LC Circuits

Given a circuit consisting of a capacitor C, an inductor L, and a switch. Find the current through the circuit as a function of time as the switch is closed.

Same thing as we have done in RC and RL circuits:

$$\frac{q}{C} - L\frac{dI}{dt} = 0$$
$$\frac{q}{C} = L\frac{d^2q}{dt^2}$$
$$\frac{1}{LC}q = \frac{d^2q}{dt^2}$$

Remember simple harmonic motion? Time to pull those out again:

$$q(t) = A\cos(\omega t + \varphi)$$

Where $\omega^2 = \frac{1}{LC}$

Depending on the starting conditions, whether the capacitor or the inductor is fully charged in the beginning, the phase shift changes to accommodate that.

Conclusion

Monday, August 17, 2020 23:05

For you, this might be close to the end of the year, cramming the last little bit of information, doing every single possible past paper, familiarizing yourself with all the concepts mentioned. To that, I wish you good luck.

For you, this might be the start of a school year, where you are anxiously anticipating a world of physics currently beyond your grasp, and you are diligently previewing the material that you are about to be presented in the coming year. To that, I wish you a safe journey through the course.

For you, this might be nostalgic. You might be sitting down in front of an ancient device, browsing a document in a format long abandoned, remembering details that have been long forgotten. To that, I hope this invoked happy memories.

For you, this might be an exemplar. You might be doing science that is far and beyond the scope of what I can dream of as I am writing this, and these notes may serve as a way to either return to the fundamentals, or a method for grasping the natural flow of learning physics.

As for me, this is the end of an era. These notes serves as my last experience of high school, where I, fittingly, leave behind a stash of notes for high school students. As I venture beyond this point, I hope you had a great time learning the concepts of introductory physics.

Boris Li
17 August, 2020