Introduction

Thursday, June 25, 2020 14:45

The actual BC part of the calculus course was taught in the second half of the year, after Christmas break. The notes were not the renewed version that is probably used starting 2020, but I am confident that I can deliver a clear, restructured version of these concepts.

Boris Li
15 July, 2020

1 Series

Wednesday, July 15, 2020 10:49

The concept of series is closely linked to the integral; integration is merely a series that, instead of adding intervals of length 1, adding intervals of the length of a differential.

1.1 Concept

1.1.1 Finite series

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

1.1.2 Infinite series

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n + a_{n+1} + \cdots$$

$$S_n = \sum_{i=1}^{n} a_i$$

If $\lim_{n\to\infty} a_n$ converges/diverges, $\sum_{i=1}^{\infty} a_i$ converges/diverges.

1.1.3 Geometric

A geometric series has terms that differ by a ratio.

$$\sum_{i=0}^{\infty} ar^i = a + ar + ar^2 + ar^3 + \cdots$$

Convergence $|r| < 1$

Divergence $|r| \geq 1$

$$S = \frac{a}{1-r}$$

1.1.4 Harmonic

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

Always diverges.

1.2 Convergence Tests

Note: These tests have not been rigourously proven in AP Calculus BC.

1.2.1 n^th Term Test

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

If the series converges, the ratio must be smaller than 1, then the terms must be approaching 0. Conversely, if the terms are not approaching 0, the ratio must be at least 1, then the series must diverge.

1.2.2 Integral Test

If a function is continuous, positive, and decreasing:

$$\sum_{n=1}^{\infty} f(n) \text{ converges/diverges as } \int_1^{\infty} f(n)dx$$

$$\sum_{n=1}^{\infty} f(n) \sim \int_1^{\infty} f(n)dx$$

This works, as $\sum_{n=2}^{\infty} f(n)$ is merely the RRAM of $\int_1^{\infty} f(n)dx$.

The RRAM is always smaller than the integral, so when the integral converges, the RRAM converges.

$$\sum_{n=1}^{\infty} f(n) = RRAM + f(1)$$

Convergence does not depend on first finite number of terms.

1.2.3 p-series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Considering $p \leq 0$,

The series diverges by the n^th term test, since the n^th term is infinitely large.

Considering $0 < p < 1$,

The series is continuous, positive and decreasing within $[1,\infty)$

$$\int_1^{\infty} \frac{1}{x^p} dx$$

$$= \lim_{b\to\infty} \int_1^{b} x^{-p} dx$$

$$= \lim_{b\to\infty} \left. \frac{x^{-p+1}}{-p+1} \right|_1^b$$

$$= \lim_{b\to\infty} \frac{b^{1-p}}{1-p} - \frac{1}{1-p}$$

$$= \lim_{b\to\infty} \frac{b^{1-p} - 1}{1-p}$$

b^{1-p} goes to infinity, the series diverges.

Considering $p = 1$,

The series is continuous, positive and decreasing within $[1,\infty)$

$$\int_1^{\infty} \frac{1}{x} dx$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{1}{x} dx$$

$$= \lim_{b\to\infty} \ln b - \ln 1$$

$$= \lim_{b\to\infty} \ln b$$

$$= \infty$$

The series diverges.

Considering $p > 1$,

The series is continuous, positive and decreasing within $[1,\infty)$

$$\int_1^{\infty} \frac{1}{x^p} dx$$

$$= \lim_{b\to\infty} \int_1^b x^{-p} dx$$

$$= \lim_{b\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{(1-p)x^{p-1}}\Bigg|_1^b$$

$$= \lim_{b\to\infty} \frac{1}{(1-p)b^{p-1}} - \frac{1}{1-p}$$

$$= 0 - \frac{1}{1-p}$$

$$= \frac{1}{p-1}$$

The series converges.

1.2.4 Alternating Series Test

If $a_n \geq a_{n+1}$ and if $\lim_{n\to\infty} a_n = 0$,

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges.}$$

If the terms get smaller and approach, the alternating series will eventually zero into a value, as the partial sums oscillate around the value.

1.2.5 Ratio Test

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Like an infinite geometric series, if the ratio is smaller than 1, the series converges, and vice versa.

1.2.6 Comparison Test

Only use when positive. If $0 < a_n < b_n$

If $\sum_{n=1}^{\infty} b_n$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ diverges.

If a larger series converges, the smaller one must also converge.
If a smaller series diverges, the larger one must also diverge.

1.2.7 Limit Comparison Test

Only use when positive.

If $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ then $\sum_{n=1}^{\infty} a_n \sim \sum_{n=1}^{\infty} b_n$

Since we know the ratio between the two terms exist, they must behave similarly, and only differ by a ratio. They must share the same convergence.

1.2.8 Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ converges.

$\sum_{n=1}^{\infty} 2|a_n|$ converges.

By comparison, since $0 < a_n + |a_n| < 2|a_n|$, $\sum_{n=1}^{\infty} a_n + |a_n|$ converges.

Since $\sum_{n=1}^{\infty} a_n + |a_n|$ and $\sum_{n=1}^{\infty} |a_n|$ both converge, $\sum_{n=1}^{\infty} a_n$ must converge.

1.3 Power Series

1.3.1 Definition

$\sum_{n=0}^{\infty} cx^n$ is a power series centered at $x = 0$.

$\sum_{n=0}^{\infty} c(x-a)^n$ is a power series centered at $x = a$.

1.3.2 General Form

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

Geometric series, so convergence $|x| < 1$.

$$S = \frac{1}{1-x}$$

Conditions:

- Converge over an interval
- Always converge at the centre x=a
- Converge over all real numbers

1.3.3 Extension

Since:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Then:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$$

And:

$$\ln(1+x) = \int \frac{dx}{1+x}$$

$$\ln(1+x) = \int \sum_{n=0}^{\infty} (-x)^n \, dx$$

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$

Plus:

$$\tan^{-1} x = \int \frac{dx}{1+x^2}$$

$$\tan^{-1} x = \int \sum_{n=0}^{\infty} (-x^2)^n \, dx$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

1.4 Taylor and MacLaurin Series

1.4.1 MacLaurin Series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(0)x^k}{k!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{3!} + \cdots$$

1.4.2 Taylor Series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(a)(x-a)^k}{k!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

1.5 Convergence (for Power Series)

1.5.1 Radius of Convergence

When a power series converges below R and diverges above R, R is the radius of convergence

1.5.2 Interval of Convergence

The interval of convergence is between a-R and a+R.

1.6 Error Bounds

1.6.1 Alternating Series Bound

For series that are alternating and converge (see 1.2.4)

$$|S - S_n| \leq a_{n+1}$$

Where S_n is the partial sum, and a_{n+1} is the next term.

1.6.2 Lagrange Error Bound

For any function approximated using an n^{th} degree Taylor series (see 1.4.2)

$$|f(b) - P_n(b)| < \frac{M}{(n+1)!} |b-a|^{n+1}$$

Where M is the maximum value of $f^{n+1}(z), a \leq z \leq b$, for a Taylor series centered around $x = a$.

2 Non-Cartesian Functions

Monday, July 20, 2020 16:06

2.1 Parametric Functions

2.1.1 Definition

Parametric functions are functions that describe coordinates as a function of another variable.

$$x = f(t)$$
$$y = g(t)$$
$$a \leq t \leq b$$

t is a parameter, a is the initial point, b is the terminal point.

2.1.2 Conversion

Convert between Cartesian and Parametric by eliminating the variable t through substitution.

2.1.3 Calculus

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$
$$\frac{dx}{dt} \neq 0$$

A parametric function only has a derivative when both x and y have derivatives.

2.2 Vector Functions

Think physics.

2.2.1 Definition

Similar to parametric functions, vector functions describe a trajectory using two different functions.

$$s(t) = \langle x(t), y(t) \rangle$$
$$s(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

2.2.2 Calculus

Velocity

$$s'(t) = v(t)$$

Acceleration

$$s''(t) = a(t)$$

Speed (magnitude of velocity)

$$\vec{v}(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

2.3 Polar Functions

See Appendix 1 for graphs.

2.3.1 Definition

Polar coordinates (r, θ)

Polar functions $r = f(\theta)$

2.3.2 Conversion

$$y = r \sin \theta$$
$$x = r \cos \theta$$

2.3.3 Calculus

Slope of tangent

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{(r \sin \theta)'}{(r \cos \theta)'}$$

Area of a region

$$A = \int_{\theta=\alpha}^{\beta} \frac{1}{2} r^2 d\theta = \int_{\theta=\alpha}^{\beta} \frac{1}{2} f^2(\theta) d\theta$$

Area between curves

$$A = \int_{\theta=\alpha}^{\beta} \frac{1}{2} (r_1^2 - r_2^2) d\theta$$

3 Supplemental Concepts

Tuesday, July 21, 2020 13:13

3.1 Limits and L'Hôpital's Rule

3.1.1 Limits Evaluated as a Derivative

See Calculus AB 4.0.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

3.1.2 Limits Evaluated using L'Hôpital's Rule

See Calculus AB 5.7.

When $\frac{0}{0}$

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

When $\frac{\infty}{\infty}$

$$\frac{f}{g} = \frac{1}{g} \cdot \frac{f}{1} = \frac{\frac{1}{g}}{\frac{1}{f}}$$

When $\infty \cdot 0$

$$f \cdot g = \frac{g}{\frac{1}{f}}$$

When $\infty - \infty$

These often appear in the form of a fraction, where the denominator approaches 0.

Try multiplying the denominators together and form one fraction.

$$\frac{1}{f} - \frac{1}{g} = \frac{g - f}{fg}$$

When an exponent exists

Try taking the logarithm of both sides. It should simplify to the above scenarios.

3.2 Logistic and Exponential Growth

3.2.1 Exponential Growth

Where the growth is dependent on the quantity

i.e. the derivative is based upon the original function

$$\frac{dP}{dt} = kP$$

$$\int \frac{1}{kP} dP = \int dt$$

$$\ln|P| = kt + c$$

$$P = e^c e^{kt}$$

$$P = Ce^{kt}$$

3.2.2 Logistic Growth

Where the curve reaches a maximum

i.e. Sigmoid curve

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$
$$\frac{dP}{dt} = k\left(\frac{M-P}{M}\right)P$$
$$\int \frac{1}{P(M-P)} dP = \frac{k}{M}\int dt$$
$$\ln|P| + \ln|P - M| = kt + c$$
$$\frac{P}{P+M} = Ce^{kt}$$
$$1 + \frac{M}{P} = Ce^{-kt}$$
$$\frac{M}{P} = 1 + Ce^{-kt}$$
$$P = \frac{MCe^{kt}}{Ce^{kt} + 1}$$
$$P = \frac{M}{1 + Ce^{-kt}}$$

3.3 Euler's Method

Iterative method to approximate a value, given a derivative function

$$y(a) = b$$
$$E(a + \Delta x) = b + \left.\frac{dy}{dx}\right|_a \Delta x$$
$$E(a + 2\Delta x) = b + \left.\frac{dy}{dx}\right|_a \Delta x + \left.\frac{dy}{dx}\right|_{a+\Delta x} \Delta x$$
$$E(a + n\Delta x) = b + \sum_{k=1}^{n} \left.\frac{dy}{dx}\right|_{a+(k-1)\Delta x} \Delta x$$

3.4 Length of a Function

3.4.1 Arc Length of a Cartesian Function

$$y = f(x)$$
$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$L = \sum \sqrt{(\Delta x)^2 + (\Delta y)^2}$$
$$L = \int \sqrt{(dx)^2 + (dy)^2}$$
$$L = \int \sqrt{\frac{(dx)^2 + (dy)^2}{(dx)^2}} dx$$
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

3.4.2 Arc Length of a Parametric Curve

$$L = \int \sqrt{(dx)^2 + (dy)^2}$$
$$L = \int \sqrt{\frac{(dx)^2 + (dy)^2}{(dt)^2}} dt$$

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

3.4.3 Arc Length of a Polar Curve

$$L = \int \sqrt{(dx)^2 + (dy)^2}$$

$$L = \int \sqrt{\frac{(dx)^2 + (dy)^2}{(d\theta)^2}}\, d\theta$$

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, dt$$

$$L = \int_{\theta=a}^{b} \sqrt{((r\cos\theta)')^2 + ((r\sin\theta)')^2}\, dt$$

4 Integration Techniques

Tuesday, July 21, 2020 15:16

4.1 Improper Integrals

Integrals which involve ∞.

$$\int_a^\infty f(x)dx$$
$$= \lim_{b\to\infty} \int_a^b f(x)dx$$
$$= \lim_{b\to\infty} F(x)\Big|_a^b$$
$$= \lim_{b\to\infty} F(b) - F(a)$$

4.2 Integration by Parts

4.2.1 Reverse Product Rule

$$(uv)' = u'v + v'u$$
$$d(uv) = udv + vdu$$
$$\int d(uv) = \int udv + \int vdu$$
$$\int udv = uv - \int vdu$$

4.2.2 Tabular Integration by Parts

Also called the Tic Tac Toe method

$$\int f(x)g(x)dx$$

$f(x)$	$g(x)$	
$f'(x)$	$\int g(x)dx$	$+$
$f''(x)$	$\iint g(x)dx^2$	$-$
$\vdots$	$\iiint g(x)dx^3$	$+$
0	$\vdots$	$\vdots$

$$\int f(x)g(x)dx = f(x)\int g(x)dx - f'(x)\iint g(x)dx^2 + f''(x)\iiint g(x)dx^3 - \cdots + C$$

Tips:

1. Differentiate in this order:
   a. Logarithms
   b. Inverse trigonometric functions
   c. Power functions
   d. Trigonometric functions
   e. Exponentials
2. Integrate what you can. Differentiate the rest.

3. If nothing else works, try integrating 1.

4.3 Partial Fractions

Since

$$\frac{A}{x+a}+\frac{B}{x+b}=\frac{A(x+b)+B(x+a)}{(x+a)(x+b)}$$

Then

$$\frac{(A+B)x+Ab+Ba}{x^2+(a+b)x+ab}=\frac{A}{x+a}+\frac{B}{x+b}$$

And thus

$$\int \frac{Cx+D}{x^2+(a+b)x+ab}dx=\int\left(\frac{A}{x+a}+\frac{B}{x+b}\right)dx$$

Where

$$A(x+b)+B(x+a)=Cx+D$$
$$(A+B)x+Ab+Ba=Cx+D$$
$$\therefore \begin{cases} A+B=C \\ Ab+Ba=D \end{cases}$$

Conclusion

Tuesday, July 21, 2020 15:56

I studied the AP Calculus BC in my grade 11 year, and have fortunately gotten a 5 on the exam. Looking back, it was not fortune, or innate knowledge, or hard work that got me such results; the difficulty of Calculus BC as compared to AB was simply not wide enough, and as such, anybody who is confident enough to take the course will almost always be able to grasp the material.

Other than my own opinion, there is other evidence that suggests the difficulty of Calculus BC being slightly too low. My fellow 11th grade classmates, Larry Yu and Christian Turpin, in their senior year, have decided to take MATH 200 at the University of Victoria. On the r/APStudents subreddit, it is a common occurrence to see students drafting scores for imaginary courses: 5s for AP Minecraft, 6s for AP Procrastination, and interestingly, many of which included an AP Calculus CD course. This phenomenon indicates the students' wishes to establish a more difficult course for calculus, one that maybe encompasses the material from MATH 200 courses, or introduce proofs using epsilon notation.

However, no matter how easy for me this course was, I absolutely do not regret learning all about calculus before entering university. With these notes that I am currently finishing up, I hope you, my reader, gain a passion for studying calculus in high school, or if you are looking back at this document for revision, good luck on your exams.

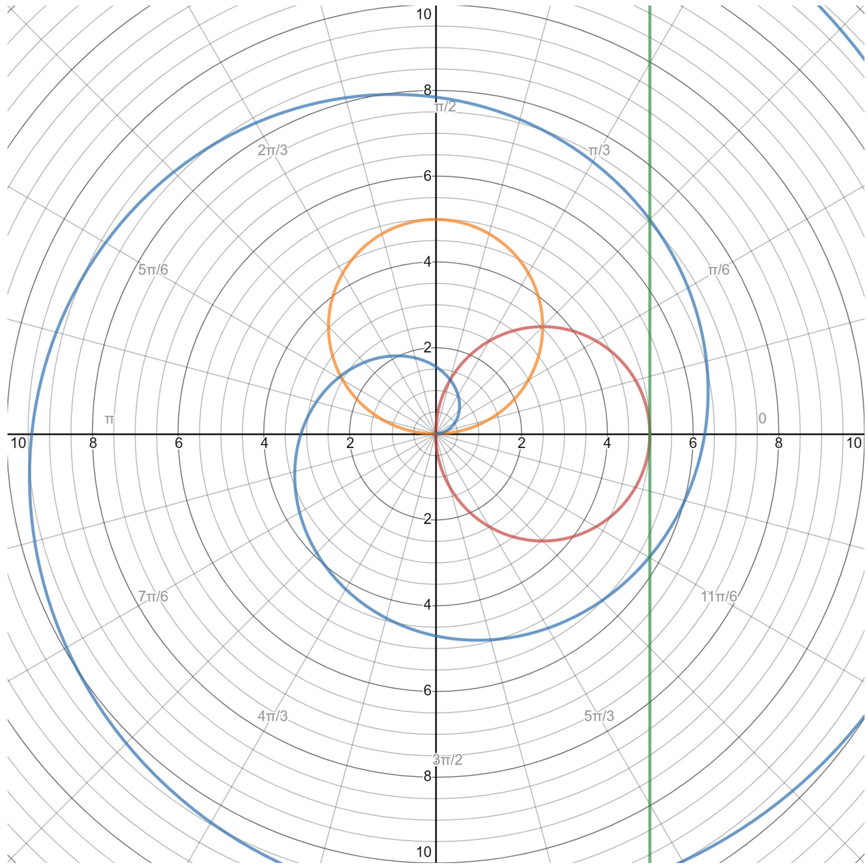
Boris Li
21 July, 2020

Appendix 1 Polar Graphs

Monday, July 20, 2020 17:12

Special Polar Graphs

Simple



Red

Circle with diameter a, right of the origin.

$r = a\cos\theta$

$0 \le \theta \le \pi$

Pictured in graph:

$r = 5\cos\theta$

Orange

Circle with diameter a, above the origin.

$r = a\sin\theta$

$0 \le \theta \le \pi$

Pictured in graph:

$r = 5\sin\theta$

Blue

Anticlockwise spiral, expands a every 360°.

$r = \frac{a}{2\pi}\theta$

$\theta \ge 0$

Pictured in graph:

$r = \theta$

Green

Vertical line at a.

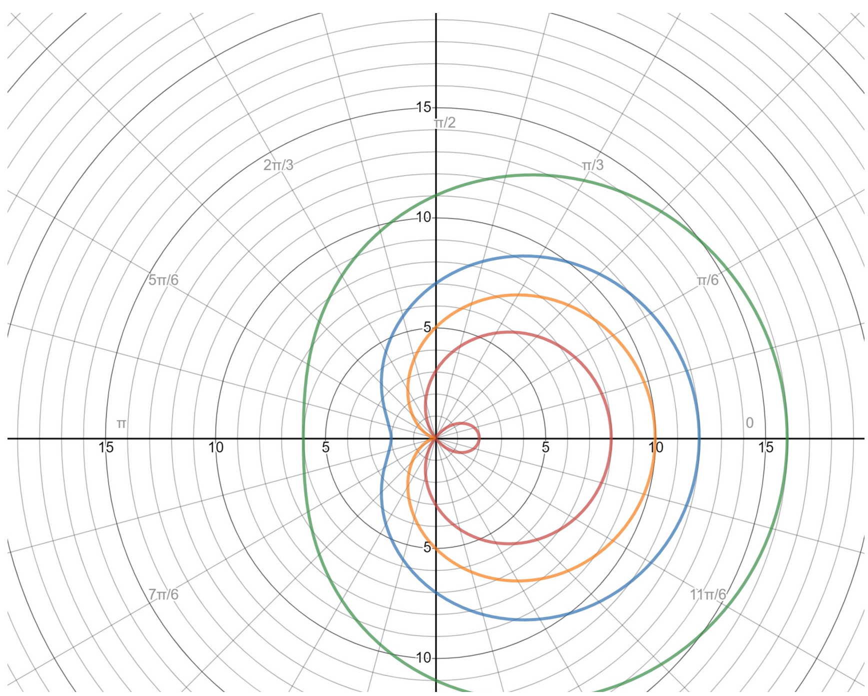
$r = \frac{a}{\cos\theta}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Pictured in graph:

$r = \frac{5}{\cos\theta}$

Limaçons



Red

Inner-loop limaçon.

Inner diameter* $b - a$

Outer diameter* $b + a$

$r = a \pm b\cos\theta$

$\left|\frac{a}{b}\right| < 1$

$0 \le \theta \le 2\pi$

Pictured in graph:

$r = 3 + 5\cos\theta$

Orange

Cardiod with diameter* $2b$.

$r = a \pm b\cos\theta$

$\left|\frac{a}{b}\right| = 1$

$0 \le \theta \le 2\pi$

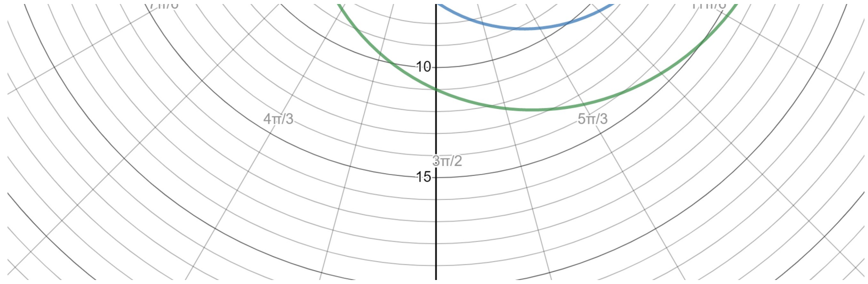
Pictured in graph:

$r = 5 + 5\cos\theta$

Blue

Pictured in graph:

$r = 5 + 5\cos\theta$



Blue

One-loop limaçon, convex from origin.
Minor radius* $a - b$
Major radius* $a + b$

$r = a \pm b\cos\theta$

$1 < \left|\frac{a}{b}\right| < 2$

$0 \le \theta \le 2\pi$

Pictured in graph:

$r = 7 + 5\cos\theta$

Green

One-loop limaçon, concave from origin.
Minor radius* $a - b$
Major radius* $a + b$

$r = a \pm b\cos\theta$

$\left|\frac{a}{b}\right| > 2$

$0 \le \theta \le 2\pi$

Pictured in graph:

$r = 11 + 5\cos\theta$

***Not proper terms**

Roses

Red

Rose with $2n$ petals and petal length of a.

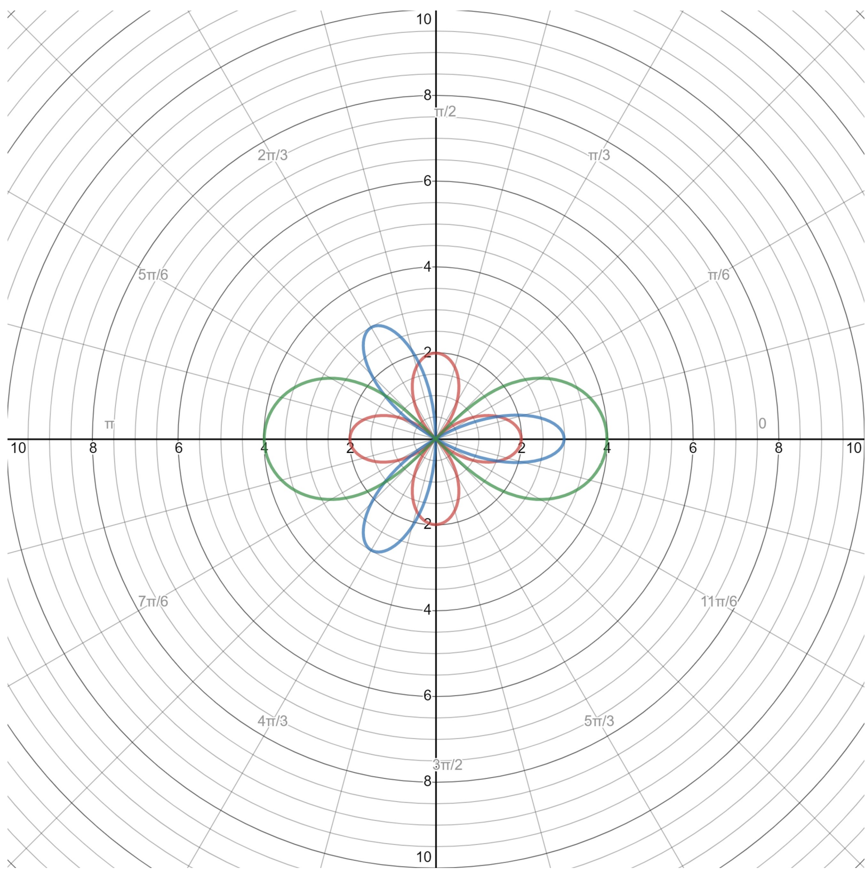
$r = a\cos n\theta$

$\frac{n}{2} \in \mathbb{Z}$

$0 \le \theta \le 2\pi$

Pictured in graph:

$r = 2\cos 2\theta$



Blue

Rose with n petals and petal length of a.

$r = a\cos n\theta$

$\frac{n}{2} + 1 \in \mathbb{Z}$

$0 \le \theta \le \pi$

Pictured in graph:

$r = 3\cos 3\theta$

Green

Lemniscate with petal length of a.

$r = a\sqrt{\cos n\theta}$

$0 \le \theta \le 2\pi$

Pictured in graph:

$r = 4\sqrt{\cos 2\theta}$

***Fractional n is definitely possible, but are not discussed in Calculus BC. In fact, a rose will have finite petals iff n is a rational number.**

Appendix 2 Integration Proofs

Tuesday, July 21, 2020 16:18

Trigonometric Functions

Tangent

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$$

$$u = \cos x$$
$$du = -\sin x \, dx$$

$$\int \tan x \, dx = \int -\frac{du}{u}$$

$$\int \tan x \, dx = -\ln|u| + C$$

$$\int \tan x \, dx = -\ln|\cos x| + C$$

Cotangent

$$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$$

$$u = \sin x$$
$$du = \cos x \, dx$$

$$\int \cot x \, dx = \int \frac{du}{u}$$

$$\int \cot x \, dx = \ln|u| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

Secant

$$\int \sec x \, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$\int \sec x \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

$$u = \sec x + \tan x$$
$$du = (\sec x \tan x + \sec^2 x) dx$$

$$\int \sec x \, dx = \int \frac{du}{u}$$

$$\int \sec x \, dx = \ln|u| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

Cosecant

$$\int \csc x\,dx = \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x}dx$$

$$\int \csc x\,dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}dx$$

$$u = \csc x + \cot x$$
$$du = (-\csc x \cot x - \csc^2 x)dx$$

$$\int \csc x\,dx = \int -\frac{du}{u}$$

$$\int \csc x\,dx = -\ln|u| + C$$

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

Inverse Trigonometric Functions

Arcsine

$$\int \sin^{-1} x\,dx$$

$$u = \sin^{-1} x$$
$$dv = dx$$

$$\int \sin^{-1} x\,dx = x \sin^{-1} x - \int \frac{xdx}{\sqrt{1-x^2}} + C$$

$$\int \sin^{-1} x\,dx = x \sin^{-1} x + \sqrt{1-x^2} + C$$

Arccosine

$$\int \cos^{-1} x\,dx$$

$$u = \cos^{-1} x$$
$$dv = dx$$

$$\int \cos^{-1} x\,dx = x \cos^{-1} x - \int -\frac{xdx}{\sqrt{1-x^2}} + C$$

$$\int \cos^{-1} x\,dx = x \cos^{-1} x - \sqrt{1-x^2} + C$$

Arctangent

$$\int \tan^{-1} x\,dx$$

$$u = \tan^{-1} x$$
$$dv = dx$$

$$\int \tan^{-1} x\,dx = x \tan^{-1} x - \int \frac{xdx}{1+x^2} + C$$

$$\int \tan^{-1} x\,dx = x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

Arccotangent

$$\int \cot^{-1} x\, dx$$

$$u = \cot^{-1} x$$
$$dv = dx$$

$$\int \cot^{-1} x\, dx = x \cot^{-1} x - \int -\frac{xdx}{1+x^2} + C$$

$$\int \cot^{-1} x\, dx = x \cot^{-1} x + \frac{1}{2}\ln(1+x^2) + C$$

Arcsecant

$$\int \sec^{-1} x\, dx$$

$$u = \sec^{-1} x$$
$$x = \sec u$$
$$dx = \tan u \sec u\, du$$

$$\int \sec^{-1} x\, dx = \int u \tan u \sec u\, du$$

$$u = u$$
$$dv = \tan u \sec u\, du$$

$$\int \sec^{-1} x\, dx = u \sec u - \int \sec u\, du + C$$

$$\int \sec^{-1} x\, dx = u \sec u - \ln|\sec u + \tan u| + C$$

$$c^2 + 1 = x^2$$
$$c = \sqrt{x^2 - 1}$$

Where c is the opposite, the adjacent measures 1, x is the hypotenuse, all in reference to the angle that measures u.

$$\int \sec^{-1} x\, dx = x \sec^{-1} x - \ln\left(|x| + \sqrt{x^2 - 1}\right) + C$$

Arccosecant

$$\int \csc^{-1} x\, dx$$

$$u = \csc^{-1} x$$
$$x = \csc u$$
$$dx = -\cot u \csc u\, du$$

$$\int \csc^{-1} x\, dx = \int -u \cot u \csc u\, du$$

$$u = u$$
$$dv = -\cot u \csc u\, du$$

$$\int \csc^{-1} x\, dx = u \csc u - \int \csc u\, du + C$$

$$\int \csc^{-1} x\, dx = u \csc u - \ln|\csc u - \cot u| + C$$

$$c^2 + 1 = x^2$$
$$c = \sqrt{x^2 - 1}$$

Where the opposite measures 1, c is the adjacent, x is the hypotenuse, all in reference to the angle that measures u.

$$\int \csc^{-1} x\, dx = x \csc^{-1} x - \ln\left(|x| - \sqrt{x^2 - 1}\right) + C$$

Logarithmic Functions

Base Case

$$\int \ln x\, dx$$

$$u = \ln x$$
$$dv = dx$$

$$\int \ln x\, dx = x \ln x - \int \frac{x}{x} dx + C$$

$$\int \ln x\, dx = x \ln x - x + C$$

General Case

$$\int \ln ax\, dx$$

$$u = \ln ax$$
$$dv = dx$$

$$\int \ln x\, dx = x \ln ax - \int \frac{x}{x} dx + C$$

$$\int \ln x\, dx = x \ln ax - x + C$$

General Logarithms

$$\int \log_b x\, dx = \int \frac{\ln x}{\ln b} dx$$

$$\int \log_b x\, dx = \frac{1}{\ln b} \int \ln x\, dx$$

$$\int \log_b x\, dx = \frac{1}{\ln b} (x \ln x - x) + C$$

$$\int \log_b x\, dx = x \log_b x - \frac{x}{\ln b} + C$$