Stability of Two-fluid Concentration Gradients

by

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Abstract

We present an energy stability analysis of different concentration gradients and mixtures of two potentially non-Newtonian fluids in a pipe, with different rheological properties such as density and viscosity, which allows us to predict its different regimes of stability. We specifically differentiate between the cases of finite-time stability, infinite-time stability, and unstable systems, and establish mathematical criteria for finite-time stability in the case of two Bingham fluids. Of the two criteria, we show that there exist physical conditions where one of them holds true.

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List of Symbols and Notation

Vector Calculus

a	Dimensionless quantity
$\hat{a}$	Quantity with dimensions
$\mathbf{a}$	Tensors of rank ≥ 1, such as vectors or matrices
$\mathbf{a}_i, \mathbf{a}_{ij}$	Components of vectors, matrices, and tensors
$\mathbf{a} \cdot \mathbf{b}$	Inner product of two vectors (rank 1 tensors), equivalent to $\sum_i a_i b_i$
$\mathbf{a} : \mathbf{b}$	Inner product or contraction of two rank 2 tensors, equivalent to $\sum_{ij} a_{ij} b_{ij}$
δ^i_j, δ_{ij}	Kronecker delta
ϵ_{ijk}	Levi-Civita symbol

Integral Calculus

Ω	Domain of concern
$\partial\Omega$	Boundary of domain
Λ_T, Λ_B	Top and bottom of domain
$\|a\|_p, \|a\|_{p,\Omega}$	L^p norm (on Ω)
$\|a\|_2, \|a\|$	L^2 norm
$\|a\|_{k,p}, \|a\|_{k,p,\Omega}$	$W^{k,p}$ norm (on Ω)
$\langle a \rangle$	Average value or integral over a domain

Differential Calculus

$\frac{\mathrm{d}}{\mathrm{d}t}$	Derivative
$\frac{\partial}{\partial t}$	Partial derivative
$\frac{\mathrm{D}}{\mathrm{D}t}$	Material derivative
∇	Del operator
$\boldsymbol{\nabla}^\alpha$	Partial derivative with respect to a multi-index α

Physical Quantities

L	(Characteristic) length
x, y, z	Cartesian coordinates
$\mathbf{u}, u, v, w$	Velocity field and components
t	Time
c	Concentration (of fluid A)
$\rho, \hat{\rho}_A, \hat{\rho}_B$	Density (in general, and of fluids A and B respectively)
p	Pressure
$\mu, \mu_{\min}$	Viscosity and minimum viscosity
g	External or gravitational acceleration
D	Diffusivity
$\boldsymbol{\gamma}$	Shear strain
$\dot{\gamma}, \dot{\boldsymbol{\gamma}}$	Shear rate value or tensor
$\tau, \boldsymbol{\tau}$	(Deviatoric) shear stress value or tensor
$\tau_Y, \tau_{\min}$	Yield stress and minimum yield stress
Re	Reynolds number
Pe	Péclet number
$\mathcal{K}$	Kinetic energy
$\mathcal{U}$	Potential energy
$\mathcal{E}$	Total energy of system

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Chapter 1

Introduction

Since antiquity, humanity has always gathered around bodies of water, as it is an essential element to the livelihoods of people. To name just a few inventions, as a civilization, we have engineered means to irrigate farmlands, erected dams to redirect flows of rivers, and utilized the abunance and properties of water to generate and store energy.

In developing these technologies, simple principles were discovered via experiments. Buoyancy was quantified by Archimedes in the equation $F = \rho g V$, relating force to density and volume of objects. Later, in the 18th century, Bernoulli and Euler derived the relationship between pressure and speed of a fluid, culminating in Bernoulli's equation, which postulates that $p + \frac{1}{2}\rho v^2 + \rho g h$ is constant.

As we approached modern times, attempts were made to describe the flow of fluids in detail using the language of differential equations. The works of Navier and Stokes allows a formulation of the equations of motion of Newtonian fluids taking into account their stresses, viscosities, and other properties. Combined with the theories and equations that describe other physical phenomena, we can come to a description of the flow of any material (provided that it *does* flow).

Aside from its physical significance, the Navier-Stokes equations also bear mathematical interest. As the equations themselves are nonlinear, the solutions (which represent the flow) are often difficult to find. In fact, it remains an open problem to demonstrate whether these equations always admit smooth solutions in three dimensions [Fefferman, 2000], and this question was posed by the Clay Mathematics Institute as one of the seven Millenium Prize problems. [Tao, 2007] remarks that this problem is difficult, and without a major breakthrough in mathematics, it might be impossible to tackle this problem.

Nevertheless, even though our current theory of partial differential equations does not always give us analytic solutions, numerical techniques are often employed to find approximate solutions. In the advent of high computational power, computational fluid dynamics becomes a feasible method of simulating realistic fluid flow up to a negligible amount of error, and as a result is widely used in various fields of engineering, such as aerodynamics and hydrodynamics.

Throughout this thesis, we will focus our attention on a small subclass of fluids, and study the time evolution of systems with continuous gradients of fluid properties. In particular, we will not provide an exact solution to the set of Navier-Stokes equations, but rather inspect the resulting energy of the whole system, and how such a quantity decays over time.

In the upcoming chapter, we will introduce *ab initio* the relevant components of fluid mechanics, and combine them to write down the Cauchy momentum equation, which forms the starting point for describing the flow of all fluids. We will then venture into the defining characteristics of Newtonian and non-Newtonian fluids, and give a few examples of mathematical models of such fluids. To conclude the chapter, we will briefly mention the applications and results of fluid gradients, and introduce a method for exploring these stability problems. We then take a detour in Chapter 3, where we formally define spaces of functions, and prove several useful properties. From these definitions, we will then introduce three classes of differential and integral inequalities, which will come into use when we formulate our physical problem. In Chapter 4, we will start from the Cauchy momentum equation and derive the set of Navier-Stokes equations that are applicable to our setup.

From there, we will transform those equations into energy equations, which will then become energy inequalities when we apply upper bounds to each of the terms. We will replicate the infinite-time decay results for Newtonian fluids, and then present a new result on finite-time decay for Bingham fluids in two different approaches. We then in turn justify the former approach, before attempting to make sense of the latter approach. Finally, we conclude in Chapter 5 by summarizing our results and discuss possible future work.

Chapter 2

Theory

2.1 Introduction to Fluid Mechanics

In this section, we shall derive, from first principles, the equations of motion governing fluid flow. We shall loosely follow the derivations in [Batchelor, 1967] and [Leal, 2007], which are introductory fluid mechanics textbooks for applied mathematicians and engineers respectively.

2.1.1 Continuum Approximation

As we know from classical mechanics, any physical system is described completely by the position and momenta of every particle in the system. Since the size of a particle in a fluid is much smaller than the typical size of a container, tracking every single particle is highly impractical, so similarly to the motivation behind statistical mechanics, we must develop a different description of our system.

However, we can use this exact property to our advantage. We can instead keep track of a macroscopic property, the average velocity over some volume:

$$\mathbf{u} = \frac{1}{V}\int_V \mathbf{v}\,\mathrm{d}V\,. \tag{2.1}$$

This is the continuum approximation, which essentially measures the velocity at a point in space; to ensure that we can both describe variations in the velocity field in our domain and still be able to average over a large number of particles, we require our volume to be much smaller than the domain, but also much larger than the size of a particle. This approximation allows us to treat the properties of a fluid as continuous functions of position and time.

Notice this immediately gives us a boundary condition. Suppose $\mathbf{n}$ is the unit normal of a surface. Then since the net flux of in that direction must be zero, we can conclude that $\mathbf{u}\cdot\mathbf{n} = 0$.

2.1.2 Material Derivative

Now, we have completely stopped tracking the positions of each particle. For any physical property F that depends on the spatial and temporal coordinates, notice that position $\mathbf{x}$ itself is also dependent on time t. We must account for this change in particle positions as we take our derivative with respect to the material, so we have

$$\frac{\mathrm{D}F}{\mathrm{D}t} = \sum_i \frac{\partial F}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} + \frac{\partial F}{\partial t} = \sum_i u_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t} = (\mathbf{u}\cdot\boldsymbol{\nabla})F + \frac{\partial F}{\partial t} \tag{2.2}$$

where we define our operator here to be the material derivative.

2.1.3 Continuity Equation

In fluids, under normal conditions, our velocities will never be relativistic; assuming that there is no additional fluid added to the system, the total mass inside our system is a conserved quantity.

Under some representative volume V, the mass inside that volume can be represented as $\int_V \rho \, \mathrm{d}V$, where ρ is the density of the fluid. However, due to fluid potentially entering and exiting this volume according to velocity $\mathbf{u}$, following conservation of mass, the change of mass inside V must be

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V = - \int_{\partial V} \rho \mathbf{u} \cdot \mathrm{d}\mathbf{A} \,. \tag{2.3}$$

We apply the divergence theorem to obtain

$$\int_V \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) \, \mathrm{d}V = 0. \tag{2.4}$$

But since this relation holds for all volumes V inside some domain Ω, the integrand itself must be

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \tag{2.5}$$

This is the continuity equation, which is a constraint that we set on the velocity field $\mathbf{u}$.

Rewriting this equation, we can obtain

$$\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t} + \boldsymbol{\nabla} \cdot \mathbf{u} = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla}) \rho \right) + \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{2.6}$$

We say that a fluid is incompressible when the density is not affected by changes in pressure, which can be written as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \tag{2.7}$$

Under the assumption of incompressibility, our continuity equation simplifies to

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{2.8}$$

2.1.4 Stress and Pressure

Up until now, our discussion has mostly been limited to kinematics, and to fully develop a theory of dynamics, we must talk about forces. External forces can be dealt with in the usual way, by simply integrating the term across our domain Ω. However, internal forces, such as particle-particle interactions, must be quantified differently. If we assume that these forces only act on a very thin layer, then it must be possible to write this in the form

$$\mathrm{d}\mathbf{F} = \boldsymbol{\Sigma} \, \mathrm{d}A \,, \tag{2.9}$$

where $\boldsymbol{\Sigma}(\mathbf{n}, \mathbf{x}, t)$ is a vector, called the stress, dependent on the direction that it is acting (the unit norm $\mathbf{n}$ of the surface A), and of course, the spatial and temporal coordinates $\mathbf{x}, t$.

Let us consider a small tetrahedral volume, with three of its faces lying on the coordinate planes, and that the volume lies in the first octant, as in Figure 2.1. We assume that our volume is sufficiently small that position dependence is negligible. The surfaces are denoted $\mathrm{d}A_x \,, \mathrm{d}A_y \,, \mathrm{d}A_z \,, \mathrm{d}A$, with unit normals $-\mathbf{x}, -\mathbf{y}, -\mathbf{z}, \mathbf{n}$ respectively; under projections onto the coordinate planes, they are related by

$$\mathrm{d}A_x = \mathbf{x} \cdot \mathbf{n} \, \mathrm{d}A \,, \quad \mathrm{d}A_y = \mathbf{y} \cdot \mathbf{n} \, \mathrm{d}A \,, \quad \mathrm{d}A_z = \mathbf{z} \cdot \mathbf{n} \, \mathrm{d}A \,. \tag{2.10}$$

so each Cartesian component of the net force on the volume can be written as

$$(\Sigma_i(\mathbf{n}) - (x_j \Sigma_i(\mathbf{x}) + y_j \Sigma_i(\mathbf{y}) + z_j \Sigma_i(\mathbf{z})) n_j) \, \mathrm{d}A \,.^1 \tag{2.11}$$

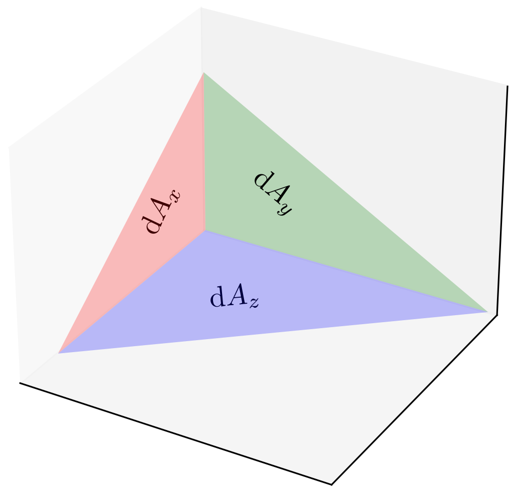

Figure 2.1: Tetrahedral volume, with three of its surfaces lying on coordinate planes.

To consider all the forces acting on our volume, we must add up the external and internal forces. But both the net and external forces scale with volume, and internal forces scale with area; if we were to let the side length ℓ go to zero, the terms of order ℓ^2 (area-dependent terms) and ℓ^3 (volume-dependent terms) must each go to zero separately. Then this allows us to write each component of our stress as the sum of three directed components. Hence, stress is a second-order tensor $\boldsymbol{\sigma}$, obeying

$$\Sigma_i = \sigma_{ij} n_j. \tag{2.12}$$

In the previous paragraphs, we have essentially considered conservation of linear momentum. We can use conservation of angular momentum to further show that there is an extra restriction on the form of the stress tensor $\boldsymbol{\sigma}$. With respect to some arbitrary point that we set to be our origin, the torque exerted on some representative volume V around this point can be represented as

$$\mathrm{d}(\mathbf{r} \times \mathbf{F}) = \epsilon_{ijk} x_j \Sigma_k \,\mathrm{d}A = \epsilon_{ijk} x_j \sigma_{km} n_m \,\mathrm{d}A\,, \tag{2.13}$$

integrated around the entire surface of our volume ∂V, where ϵ_{ijk} is the Levi-Civita symbol.[2]

We again apply the divergence theorem, to yield

$$\int_{\partial V} \mathrm{d}(\mathbf{r} \times \mathbf{F}) = \int_V \epsilon_{ijk} \frac{\partial (x_j \sigma_{km})}{\partial x_m} \,\mathrm{d}V = \int_V \epsilon_{ijk} \left(\delta_{jm} \sigma_{km} + r_j \frac{\partial \sigma_{km}}{\partial x_m} \right) \mathrm{d}V\,. \tag{2.14}$$

Analyzing dimensions, torque scales with ℓ^4, while the two terms inside our integral scales with ℓ^3 and ℓ^4 respectively. Again letting $\ell \to 0$, we have

$$\epsilon_{ijk} \delta_{jm} \sigma_{km} = 0 \implies \sigma_{kj} = \sigma_{jk}; \tag{2.15}$$

[1]We adopt two conventions here: (1) the subscripts here (such as a_j) denote components of a tensor, and in general, we do not need to distinguish between covariant and contravariant indices; and (2) we use Einstein summation notation, where if the same index appears twice, we are to assume that it is summed over all possible values of that index, such as $a_i b_i = \sum_{i=1}^{3} a_i b_i$ (in most of our applications, we are summing over the three spatial dimensions).

[2]The Levi-Civita symbol $\epsilon_{i_1 i_2 \ldots i_n}$ is defined to be the sign of the permutation $(i_1 i_2 \ldots i_n)$ if $i_1, i_2, \ldots, i_n$ are pairwise distinct, and zero otherwise.

that is, our stress tensor is symmetric.

It is well-known from linear algebra that the eigenvalues of a real symmetric matrix are all real, and that the trace is the sum of the diagonal, which is also the sum of the eigenvalues. Suppose $\tilde{\sigma}_{ij}$ denotes the components of the stress tensor in a diagonalized basis. Then our two facts from linear algebra yield $\sigma_{ii} = \tilde{\sigma}_{ii}$.

When a fluid is at rest, by spherical symmetry, the stress should be the same in every direction, so every eigenvalue should be the same. Hence, it is convenient for us to define the pressure (which is compressive in general) at a point to be

$$p = -\frac{1}{3}\sigma_{ii}. \tag{2.16}$$

However, when the fluid is moving, we can no longer expect the stress to be purely isotropic, so there can be an extra component that describes this deviation from isotropic conditions. We therefore write

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{2.17}$$

where $\boldsymbol{\tau}$ is termed the deviatoric stress tensor.

2.1.5 Diffusion

As we know from thermodynamics, systems tend to approach equilibrium; the same phenomenon happens in fluids. Suppose that we have some concentration c dependent on $\mathbf{x}, t$ that is not uniform throughout, but instead varies continuously in our domain. The concentration flux through some area A can then be written as $\mathrm{d}\Phi = \mathbf{J} \cdot \mathrm{d}\mathbf{A}$ where $\mathbf{J}$ is some vector field dependent on $\mathbf{x}, t$, using an argument similar to our derivation of the stress tensor.

The first assumption we make is that the value of $\mathbf{J}$ at any position $\mathbf{x}_0$ is only affected by the values of c and $\boldsymbol{\nabla} c$ at $\mathbf{x}_0$, and not higher spatial derivatives of concentration; this happens when every second spatial derivative is sufficiently small, which is empirically true near equilibrium. The second assumption we make is that the value of $\mathbf{J}$ depends only linearly on $\boldsymbol{\nabla} c$, which is valid when the concentration varies very gradually. Then since there is no diffusion if $\boldsymbol{\nabla} c = \mathbf{0}$, we can write the flux vector as

$$\mathbf{J} = -k\boldsymbol{\nabla} c \tag{2.18}$$

where k is some coefficient. We are allowed to assume that k itself is not tensorial, because fluids are usually isotropic, so the flux should be proportional to the gradient.

Now, the change in total number of particles according to our concentration can be represented as

$$\frac{\mathrm{D}}{\mathrm{D}t}\int_V Nc\,\mathrm{d}V - \int_{\partial V} k\boldsymbol{\nabla} c \cdot \mathrm{d}\mathbf{A} = \frac{\mathrm{D}}{\mathrm{D}t}\int_V Nc\,\mathrm{d}V - \int_V \boldsymbol{\nabla} \cdot (k\boldsymbol{\nabla} c)\,\mathrm{d}V = 0 \tag{2.19}$$

where N is some number density of molecules. Again using the same argument about this relation holding for all volumes allows us to conclude that

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} c = \boldsymbol{\nabla} \cdot (D\boldsymbol{\nabla} c) \tag{2.20}$$

where $D = k/N$ is the diffusivity of our fluid. In most cases, D is assumed to be small enough that we can approximate it to be constant, so the diffusion equation is

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} c = D\nabla^2 c. \tag{2.21}$$

2.1.6 Cauchy Momentum Equation

Newton's second law relates the change in momentum with the net force applied onto a body of matter. For some representative volume V of fluid, the momentum is given by $\int_V \rho\mathbf{u}\,\mathrm{d}V$. Recall from Section 2.1.2 that we cannot simply take the time derivative, and we should instead be taking the material derivative. On the other hand, the forces on our volume must include both the external forces (which we assume consists only of gravity) $\int_V \rho\mathbf{g}\,\mathrm{d}V$, and the internal forces $\int_{\partial V} \boldsymbol{\sigma}\cdot\mathrm{d}\mathbf{A}$. Using the divergence theorem to convert this into volume integrals, we again recall the argument that since Newton's second law applies to all volumes inside our domain, the integrands themselves can be equated, which yields the Cauchy momentum equation

$$\frac{\mathrm{D}}{\mathrm{D}t}(\rho\mathbf{u}) = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \rho\mathbf{g}. \tag{2.22}$$

Alternatively, this is

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + (\mathbf{u}\cdot\boldsymbol{\nabla})(\rho\mathbf{u}) = -\boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\mathbf{g}. \tag{2.23}$$

The momentum equation, combined with the continuity equation and a constitutive relation (such as one that governs diffusive transport), allows us to derive a set of Navier-Stokes equations that govern the flow of our system under certain assumptions about the nature of the fluid itself.

2.2 Complex Fluids

In this section, we investigate the properties that differentiate between Newtonian and non-Newtonian fluid behaviour.

2.2.1 Strain Rate and Vorticity

We turn our attention to a description of the behaviour of the fluid in the neighbourhood of a point $\mathbf{x}$. Let us first take $\boldsymbol{\nabla}\mathbf{u}$, which is a second-order tensor, to be

$$(\boldsymbol{\nabla}\mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{2.24}$$

For reasons that we will later make clear, let us define the symmetric and antisymmetric components of this matrix to be

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.25}$$

such that $\boldsymbol{\nabla}\mathbf{u} = \mathbf{e} + \boldsymbol{\xi}$. We shall call $\mathbf{e}$ the rate of strain, and $\boldsymbol{\xi}$ the vorticity.

Taking a linear approximation, let us observe that the velocity at some point $\mathbf{x} + \delta\mathbf{x}$ can be written as

$$\mathbf{u} + \delta\mathbf{u} = \mathbf{u} + \boldsymbol{\nabla}\mathbf{u}\,\delta\mathbf{x} = \mathbf{u} + (\mathbf{e} + \boldsymbol{\xi})\,\delta\mathbf{x}\,. \tag{2.26}$$

It follows that the rate of change of $\delta\mathbf{x}$, which measures the rate of deformation of our material, can be computed as

$$\frac{\mathrm{D}}{\mathrm{D}t}|\delta\mathbf{x}| = \frac{\mathrm{D}}{\mathrm{D}t}(\delta\mathbf{x}\cdot\delta\mathbf{x})^{1/2} = \frac{\delta\mathbf{x}\cdot\delta\mathbf{u}}{|\delta\mathbf{x}|} = \frac{1}{|\delta\mathbf{x}|}(\delta\mathbf{x}\cdot\mathbf{e}\,\delta\mathbf{x} + \delta\mathbf{x}\cdot\boldsymbol{\xi}\,\delta\mathbf{x}). \tag{2.27}$$

But antisymmetry yields $\delta\mathbf{x}\cdot\boldsymbol{\xi}\,\delta\mathbf{x} = 0$, which implies that this rate of deformation only linearly depends on the rate of strain.

We can similarly show that vorticity governs the rotational behaviour of the fluid. Observe that

$$\boldsymbol{\xi}\,\delta\mathbf{x} = -\frac{1}{2}\,\delta\mathbf{x} \times (\boldsymbol{\nabla} \times \delta\mathbf{u}), \tag{2.28}$$

immediately giving us an expression that involves $\boldsymbol{\nabla} \times \delta\mathbf{u}$, which measures the rotation at our point.

2.2.2 Newtonian Fluids

We shall now demonstrate the relationship between the deviatoric shear stress $\boldsymbol{\tau}$ and the rate of strain tensor $\mathbf{e}$. We apply the same assumptions as our discussion of diffusion in Section 2.1.5, and only take linear contributions of $\boldsymbol{\nabla}\mathbf{u}$. Then in the most general form, we have

$$\tau_{ij} = \mu_{ijkm}(\boldsymbol{\nabla}\mathbf{u})_{km}, \tag{2.29}$$

where $\boldsymbol{\mu}$ is a fourth-order tensor, representing some coefficient of proportionality.

Assuming that the fluid is isotropic (as we have done so throughout), the form of $\boldsymbol{\mu}$ is restricted to

$$\mu_{ijkm} = \mu\delta_{ik}\delta_{jm} + \mu'\delta_{im}\delta_{jk} + \mu''\delta_{ij}\delta_{km}, \tag{2.30}$$

where μ, μ', μ'' are scalar values. Immediately, since $\boldsymbol{\tau}$ is symmetric, we see that μ_{ijkm} must be symmetric with respect to i and j, so $\mu = \mu'$. Hence, we have

$$\tau_{ij} = \mu_{ijkm}e_{km} + \mu_{ijkm}\xi_{km} = \mu(e_{ij} + e_{ji}) + \mu''\delta_{ij}e_{kk} + \mu_{ijkm}\xi_{km} = 2\mu e_{ij} + \mu''\delta_{ij}e_{kk}. \tag{2.31}$$

and the $\boldsymbol{\xi}$ term vanishes since μ_{ijkm} is symmetric with respect to m and k too.

Now recall from our discussion of stress that by definition of the deviatoric stress tensor, $\boldsymbol{\tau}$ is traceless, so we have the restriction

$$0 = \tau_{ii} = 2\mu e_{ii} + 3\mu'' e_{kk} = (2\mu + 3\mu'')\boldsymbol{\nabla}\cdot\mathbf{u}. \tag{2.32}$$

Solving this equation for μ'' allows us to determine that the deviatoric shear stress must be

$$\boldsymbol{\tau} = 2\mu\left(\mathbf{e} - \frac{1}{3}\boldsymbol{\nabla}\cdot\mathbf{u}\right), \tag{2.33}$$

The tensor $\boldsymbol{\mu}$ simplifies down to a single scalar constant of proportionality μ, which we shall call viscosity.

Under incompressibility, $\mathbf{u}$ is divergence-free, so the above equation simplifies to

$$\boldsymbol{\tau} = 2\mu\mathbf{e} = \mu\dot{\boldsymbol{\gamma}} \tag{2.34}$$

where we define $\dot{\boldsymbol{\gamma}} = 2\mathbf{e}$ to be the engineering strain rate, simplifying our notation. This linear relationship is called Stokes' stress constitutive equation, or sometimes Newton's law of viscosity.

2.2.3 Non-Newtonian Fluids

Throughout our derivation above, we have consistently used the assumption that our fluid is isotropic, and this assumption only holds for a small class of fluids. If we perhaps were to have a suspension of polymers dissolved in some liquid, then at the microscopic level, our fluid gains additional structure, and the derivation above no longer holds.

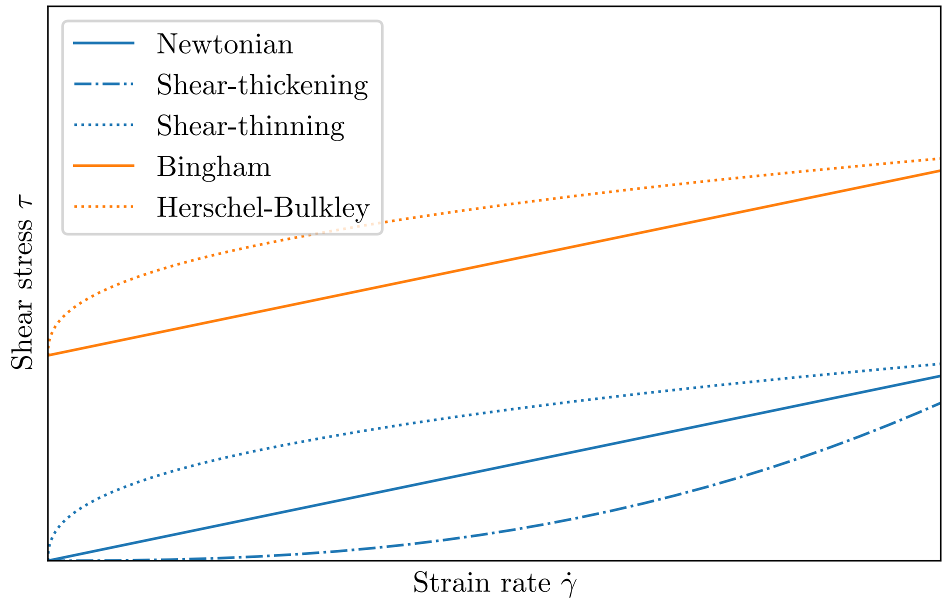

Figure 2.2: Stress-strain curves of different models of non-Newtonian fluids, with a Newtonian fluid included as a comparison. The slopes of the curves represent the apparent viscosity, which is dependent on the strain rate.

In particular, the linear relationship between shear stress and strain rate that defines a Newtonian fluid no longer holds. We shall define any fluid with such deviation from this relationship as non-Newtonian (see Figure 2.2 for examples).

From experimental evidence, it is often convenient to describe a fluid by an apparent viscosity μ, where μ is a function of a number of variables [Barnes, 1999]. There are three main effects that we are concerned with:

(i) a nonzero yield stress τ_Y initially required to move the fluid from rest;
(ii) viscosity as a function of strain rate; and
(iii) viscosity as a function of time since stress applied.

Fluids that exhibit the first of such effects are often termed yield stress fluids. They were first studied extensively by [Bingham, 1922], after whom the infamous Bingham model is named. He modeled fluids as having an unyielded state, when strain rate $\dot{\gamma} = 0$, and a yielded state, when the stress $\tau \geq \tau_Y$, where viscosity of the fluid is constant:

$$\tau = \tau_Y + \mu\dot{\gamma}. \tag{2.35}$$

This model of having an unyielded and a yielded state of a fluid was generalized by [Casson, 1959] and [Herschel and Bulkley, 1926], which gives us the two following models respectively:

$$\sqrt{\tau} = \sqrt{\tau_Y} + \sqrt{\mu\dot{\gamma}}, \tag{2.36}$$

$$\tau = \tau_Y + k\dot{\gamma}^n. \tag{2.37}$$

The mathematical formulation of extending these yield stress models to three-dimensional domains was introduced by [Hohenemser and Prager, 1932] and [Oldroyd, 1947], taking into account that stress appears as a tensorial quantity.

However, due to constraints in experiments, we can only ever measure down to some finitely small strain rate, so yield stress values are always interpolated. [Reiner, 1943] remarks that in reality, perhaps "there is no yield point"; while yield stress is a useful tool for characterizing a fluid, ultimately it only holds in the experimental ranges, and extrapolation might not always conincide with physical phenomena. This invites recent developments in the theory of rheological physics, such as the model proposed by [Kamani et al., 2021], which circumvents this piecewise description of the yielded and unyielded regions, by describing stress as a smooth function of both strain γ and strain rate $\dot{\gamma}$.

2.3 Two-Fluid Dynamics

In this section, we discuss the last major component of our problem: the appearance of a second fluid in our system. The majority of well-known results in this field lie in immiscible fluids, which are discussed extensively in [Joseph and Renardy, 2013a, Joseph and Renardy, 2013b]. We shall mainly focus on miscible fluid flow in this section.

2.3.1 Fluid Gradients

Continuing our discussion on non-Newtonian fluids, we observe that there are a wide variety of usages of such fluids in engineering and physical applications. The most prominent examples are mixtures of cement slurry, where gravel and sand (solids) are mixed with water (liquid) to form a suspension that exhibits shear-thickening behaviour. Such mixtures of materials invite us to study the flow in fluid gradients, alongside with the gradients of different rheological properties. For example, when pouring cement into water, the two compounds are miscible, but have different viscosities. As such, there is a practical application in understanding viscosity-driven flow.

We first remark that the physics governing the flow of gradients is identical to what we derived above, provided that we make the same assumptions about the properties and the nature of our fluid [Joseph and Renardy, 2013a].

Then, since any such gradient must have a physical property that obeys a constitutive equation, any analysis of these gradients must have similarities in their derivations. A general treatment of stability in temperature gradients was done by [Gribov and Gurevich, 1956], while the two papers of [Batchelor and Nitsche, 1991, Batchelor and Nitsche, 1993] investigated the stability in temperature gradients. More specifically, linear stability analyses have been done on temperature gradients [Zhang et al., 2006] and density gradients [Graf et al., 2002]. Following in their footsteps, a linear stability analysis can be performed on viscosity gradients.

2.3.2 Hydrodynamic Stability

Due to the fact that obtaining an exact solution to our problem might be prohibitively difficult, one way to understand our physical system is via probing the solutions around our point of stability. We can locally perturb this stable solution, and calculate whether these perturbations will grow with respect to time (unstable), exponentially decay (stable in infinite time), or vanish in finite time (stable in finite time).

In our problem, which is described in Chapter 4, we will be working with the stability of Bingham fluids. However, we remark that in the case of three dimensions, the hydrodynamic stability problem has always led to stability in infinite time, unlike the case of two dimensions, which is stable in finite time.

Chapter 3

Mathematical Tools

This chapter is dedicated to introducing definitions and proving a set of well-known mathematical theorems about vector fields for use later in our physical problem.

3.1 Function Spaces and Norms

We shall first discuss the spaces and properties of the functions that we will be working with, and establish a framework for which we can compare these functions. We assume that the reader is familiar with the definition of a metric, which is a notion of distance. A norm, which is sometimes induced by such a metric, allows us to give objects that are not numbers (such as functions) some notion of magnitude.[3] We specifically recall that two norms equivalent if they measure and order elements the same way; two norms $\|\cdot\|_{a,b}$ are equivalent if there exists $c, C > 0$ such that $c\|\cdot\|_a \leq \|\cdot\|_b \leq C\|\cdot\|_a$.

3.1.1 L^p Space

Let $f \in C^0(\Omega)$ be a bounded continuous function on a domain Ω. The L^p norm of f for some finite $p \geq 1$ is defined as

$$\|f\|_p = \left(\int_\Omega |f|^p \,\mathrm{d}\mathbf{x}\right)^{1/p}. \tag{3.1}$$

We can define the L^∞ norm, or the supremum norm of f to be

$$\|f\|_\infty = \sup_\Omega |f|. \tag{3.2}$$

It is an elementary exercise in real analysis to show that the two definitions above are norms. Furthermore, we can justify naming the supremum norm as the L^∞ norm with the following proposition.

Proposition 3.1. *The supremum norm is the limit of L^p norms as p approaches infinity.*

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p \tag{3.3}$$

Proof. We consider only the case of continuous functions. Let $M = \sup_{x\in\Omega} |f(x)|$. We can always bound the integral by its supremum, which gives us

$$\|f\|_p^p \leq M^p \mu(\Omega) \implies \|f\|_p \leq M(\mu(\Omega))^{1/p}$$

where $\mu(\Omega)$ denotes the measure of our domain.

[3] If the reader is unfamiliar with such concepts, we direct the reader to undergraduate textbooks in real analysis for a thorough discussion of the topic, perhaps Chapters 2 & 7 of [Rudin, 1976] and Chapters 4 & 5 of [Folland, 2013].

By continuity, $|f|$ must attain the supremum M at some point $x_0 \in \Omega$. For some arbitrary ϵ, there exists a δ, such that within a measure of size δ that contains x_0,

$$\|f\|_p^p \geq \int_\delta |f|^p \, d\mathbf{x} \geq (M-\epsilon)^p \delta \implies \|f\|_p \geq (M-\epsilon)\delta^{1/p}$$

As we let $p \to \infty$, $\mu(\Omega)^{1/p}$ and $\delta^{1/p}$ both go to 1. Hence, our inequalities become

$$M - \epsilon \leq \lim_{p\to\infty} \|f\|_p \leq M \tag{3.4}$$

Since our choice of ϵ is arbitrary, $\lim_{p\to\infty} \|f\|_p = M$ as desired. □

We note that in general, since not all (Lebesgue) integrable functions are continuous, the space of all L^p functions is much larger than the space of continuous functions C^0. The supremum norm will be defined slightly differently in that that case, with us taking the supremum over almost everywhere in Ω, $\|f\|_\infty = \operatorname{ess\,sup}_\Omega |f|$, but the idea remains the same.

To finish our quick discussion on L^p norms, we have a rough bound for the product of two norms.

Theorem 3.2. *Suppose we have two functions $f, g \in L^p$. Then the following inequality holds.*

$$\|f\|_p \|g\|_p \leq \frac{1}{2}\Big(\|f\|_p^2 + \|g\|_p^2\Big) \tag{3.5}$$

Proof. Observe that we have

$$0 \leq \Big(\|f\|_p - \|g\|_p\Big)^2 = \|f\|_p^2 - 2\|f\|_p\|g\|_p + \|g\|_p^2 \tag{3.6}$$

and adding $2\|f\|_p\|g\|_p$ to both sides yields our desired inequality. □

We shall reserve the notation $\|a\|$ to denote the L^2 norm $\|a\|_2$. Mathematical details regarding L^p spaces can be found in [Rudin, 1986], Chapter 3.

3.1.2 Sobolev Space

We can extend the notion of function spaces to include derivatives. This idea is occasionally useful when dealing with differential equations, since we often want our solution functions, along with their derivatives, to have finite norms.

Let $W^{k,p}(\Omega)$ denote the space of functions f where all (weak) partial derivatives up to kth order of f are in $L^p(\Omega)$. The $W^{k,p}$ norm of f for some finite $p \geq 1$ is defined as

$$\|f\|_{k,p} = \left(\sum_{|\alpha|\leq k} \|\boldsymbol{\nabla}^\alpha f\|_p^p\right)^{1/p} \tag{3.7}$$

where we are summing over all possible ith partial derivatives, $i \leq k$. Again, the case of $p = \infty$ is similarly defined using the L^∞ norm.

$$\|f\|_{k,\infty} = \max_{|\alpha|\leq k} \|\boldsymbol{\nabla}^\alpha f\|_\infty \tag{3.8}$$

For a more detailed discussion of the mathematical rigour behind the Sobolev space, we direct the reader to [Evans, 2010], Section 5.2.

3.2 Differential & Integral Inequalities

After defining a notion of distance and magnitudes on functions, in order to apply them to differential equations and inequalities, we must develop theorems that are able to use those definitions.

3.2.1 Hölder's Inequality

One version of Hölder's inequality relates the integral of a product to the product of integrals. A reference to this is given in [Rudin, 1976] Chapter 6, Exercise 10.

Theorem 3.3 (Hölder's inequality). *Suppose f, g are integrable on some domain Ω. If $1 \leq p, q \leq \infty$ satisfy $1/p + 1/q = 1$, then*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \tag{3.9}$$

Proof. First suppose p, q finite. We prove that $uv \leq u^p/p + v^q/q$ for all $u, v \geq 0$. The case where either u or $v = 0$ is clear, so we assume $u, v > 0$. We fix v constant and inspect the function $h(u) = u^p/p + v^q/q - uv$, and it suffices to prove this function is non-negative. Observe that $\lim_{u\to\infty} h(u) = \infty$, and $h'(u) = u^{p-1} - v$, so $h'(0) = -v < 0$. This allows us to say that h is initially decreasing, but eventually has to keep increasing, so it must achieve a minimum at some $u_0 > 0$. By the first derivative test $0 = h'(u_0) = u_0^{p-1} - v$, so $u_0^p = v^{p/(p-1)} = v^q$ and $u_0 = v^{q/p} = v^{q-1}$. But the minimum is $h(u_0) = u_0^p/p + v^q/q - u_0 v = v^q/p + v^q/q - v^q = 0$, so the entire function is non-negative.

Then if $\tilde{f}$ and $\tilde{g}$ are non-negative functions, and

$$\int \tilde{f}^p \, \mathrm{d}\mathbf{x} = 1 = \int \tilde{g}^q \, \mathrm{d}\mathbf{x} \tag{3.10}$$

from the previous claim we have $\tilde{f}\tilde{g} \leq \tilde{f}^p/p + \tilde{g}^q/q$. We integrate this to get

$$\int \tilde{f}\tilde{g} \, \mathrm{d}\mathbf{x} \leq \int \frac{\tilde{f}^p}{p} + \frac{\tilde{g}^q}{q} \, \mathrm{d}\mathbf{x} = \frac{1}{p} + \frac{1}{q} = 1 \tag{3.11}$$

Now again suppose $f, g \neq 0$, since if f or $g = 0$ we are done. Substitute $\tilde{f} = |f|/\|f\|_p$ and $\tilde{g} = |g|/\|g\|_q$, which allows us to use the previous claim to show

$$\int \frac{|fg|}{\|f\|_p \|g\|_q} \, \mathrm{d}\mathbf{x} = \frac{1}{\|f\|_p \|g\|_q} \int |fg| \, \mathrm{d}\mathbf{x} \leq 1 \tag{3.12}$$

Hence we can conclude that

$$\|fg\|_1 = \int |fg| \, \mathrm{d}\mathbf{x} \leq \|f\|_p \|g\|_q \tag{3.13}$$

as desired.

We note that if $p = \infty$ and $q = 1$, then obviously $|fg| = |f||g| \leq (\sup |f|)|g|$, where we integrate to get our desired result. ☐

We can immediately apply this to obtain a general inequality about L^p norms.

Corollary 3.4. *Suppose f is integrable on some bounded domain Ω. If $1 \leq p < q \leq \infty$, then*

$$\|f\|_p \leq C\|f\|_q \tag{3.14}$$

where C is a constant dependent only on the domain Ω.

Proof. Let 1 be the function that evaluates to 1 at every point. By Hölder's inequality we can get

$$\int_\Omega |f|^p \, \mathrm{d}\mathbf{x} = \|1 f^p\|_1 \leq \|1\|_{q/(q-p)} \|f^p\|_{q/p} = \|1\|_{q/(q-p)} \left(\int_\Omega |f|^q \, \mathrm{d}x \right)^{p/q} \tag{3.15}$$

Notice $\|1\|_{q/(q-p)}$ is essentially some volumetric quantity of our domain Ω (more precisely, it is the measure of Ω under some norm) so it should depend only on Ω. Taking the pth root gives us our desired inequality. □

Although the Cauchy-Schwarz inequality predates Hölder's inequality, for our purposes, the former is merely a special case of the latter, and we may simply state it as a corollary.

Corollary 3.5 (Cauchy-Schwarz inequality). *Suppose f, g are integrable on some domain Ω. Then*

$$\left| \int_\Omega fg \, \mathrm{d}\mathbf{x} \right| \leq \left(\int_\Omega |f|^2 \, \mathrm{d}\mathbf{x} \right)^{1/2} \left(\int_\Omega |g|^2 \, \mathrm{d}\mathbf{x} \right)^{1/2} \tag{3.16}$$

Proof. Apply Hölder's inequality to $p = q = 2$. □

3.2.2 Poincaré's Inequalities

This inequality was proven by Henri Poincaré in [Poincaré, 1890] in two dimensions. We shall prove a slightly weaker version, which will be sufficient for this problem. [Mitrinović et al., 1991] gives a treatise on similar inequalities and their conditions in Chapter II, especially Section 64.

Theorem 3.6 (Poincaré's inequality). *Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded connected open set, with the boundary $\partial\Omega$ being sufficiently nice (formally, Ω is Lipschitz). If $u \in C^\infty(\Omega)$ is infinitely differentiable, then*

$$\|u - u_\Omega\|_p \leq C \|\nabla u\|_p \tag{3.17}$$

where the norm is any L^p norm when $p \geq 1$, $C > 0$ is a constant dependent on Ω and p, and u_Ω is the average value of u on Ω.

$$u_\Omega = \frac{\|u\|_1}{\|1\|_1} \tag{3.18}$$

Proof. The case where $p = \infty$ boils down to an inequality of two real numbers, which is trivially true, perhaps by the Archimedean principle (see [Rudin, 1976] Theorem 1.20a).

We now consider $1 \leq p < \infty$. Without loss of generality suppose the average value is 0. Since Ω is bounded, we integrate by parts to get

$$\int_\Omega |u|^p \, \mathrm{d}\mathbf{x} = \int_\Omega \frac{\partial x_j}{\partial x_j} |u|^p \, \mathrm{d}\mathbf{x} = -\int_\Omega p x_j |u|^{p-1} \operatorname{sgn}(u) \frac{\partial u}{\partial x_j} \, \mathrm{d}\mathbf{x} \leq C \int_\Omega |u|^{p-1} |\nabla u| \, \mathrm{d}\mathbf{x} \tag{3.19}$$

where C is a positive constant. Observe that the first term during integration by parts disappears due to the average being 0.

If we let $q = p/(p-1)$, then Hölder's inequality gives us

$$\|u\|_p^p \leq C \left\| |u|^{p-1} |\nabla u| \right\|_1 \leq C \|u\|_q^{p-1} \|\nabla u\|_p \tag{3.20}$$

which when we divide by $\|u\|_p^{p-1}$ yields our desired result. □

A more general inequality relating the L^p norm of u to the norm of its derivatives can be found in [Gagliardo, 1959] and [Nirenberg, 1959].

Theorem 3.7 (Gagliardo-Nirenberg interpolation inequality)**.** *Suppose* $1 \le q \le \infty$*,* $j, k \in \mathbb{Z}_{>0}$*, with* $j < k$*, and either*

$$\begin{cases} r = 1 \\ \frac{j}{k} \le \theta \le 1 \end{cases} \quad or \quad \begin{cases} 1 < r < \infty \\ k - j - \frac{n}{r} \in \mathbb{Z}_{\ge 0} \\ \frac{j}{k} \le \theta < 1 \end{cases} \tag{3.21}$$

If we have

$$\frac{1}{p} = \frac{j}{n} + \theta\left(\frac{1}{r} - \frac{k}{n}\right) + \frac{1-\theta}{q} \tag{3.22}$$

then for some function $u \in L^q(\mathbb{R}^n) \cap W^{k,r}(\mathbb{R}^n)$*,*

$$\left\|\boldsymbol{\nabla}^j u\right\|_p \le C\left\|\boldsymbol{\nabla}^k u\right\|_r^\theta \|u\|_q^{1-\theta} \tag{3.23}$$

where C *is a constant not dependent on* u*.*

Furthermore, if u *is defined on some bounded domain* $\Omega \subseteq \mathbb{R}^n$*, where* $u \in L^q(\Omega) \cap W^{k,r}(\Omega)$*, then*

$$\left\|\boldsymbol{\nabla}^j u\right\|_p \le C\left(\left\|\boldsymbol{\nabla}^k u\right\|_r^\theta \|u\|_q^{1-\theta} + \|u\|_\sigma\right) \le C\|u\|_{k,r}^\theta \|u\|_q^{1-\theta} \tag{3.24}$$

where $1 \le \sigma \le \infty$ *is arbitrary, and* C *is a constant dependent on* Ω *and other variables, but not on* u*.*

We shall use this inequality without proof, as the proof itself is not particularly inspiring in terms of physical meaning. We note that in the original papers, neither Gagliardo nor Nirenberg provided a complete proof for the most general inequality, and thus we direct the reader to [Fiorenza et al., 2018] for a most complete argument.

3.2.3 Grönwall's Inequalities

Grönwall's lemma is an inequality regarding the solution to a differential inequality, given in terms of the solution of the corresponding differential equation, which was first proven by Thomas Grönwall in [Gronwall, 1919]. This result is generalized by [Bellman, 1943], which we have provided a proof of below, referencing [Mitrinović et al., 1991] Chapter XII, Theorem 1.2.

Theorem 3.8 (Grönwall's lemma)**.** *Suppose* $I = \{t : t \ge a\} \subseteq \mathbb{R}$ *denotes an interval. Let* $\alpha, u \in C^0(I)$ *be continuous on* I*,* u *be differentiable on the interior* $I^\circ = \{t : t > a\}$*, with the following inequality holding for all* $t \in I^\circ$*.*

$$u(t) = u_0 + \int_a^t \alpha(s)u(s)\,\mathrm{d}s \tag{3.25}$$

Then $u(t)$ *is bounded by the following solution.*

$$u(t) \le u(a)\exp\left(\int_a^t \alpha(s)\,\mathrm{d}s\right) \tag{3.26}$$

Proof. Let us first write down the corresponding differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} \le \alpha(t)v(t) \tag{3.27}$$

It is well-known that the solutions of this differential equation take on the form

$$v(t) = v(a)\exp\biggl(\int_a^t \alpha(s)\,\mathrm{d}s\biggr) \tag{3.28}$$

Let $v(a) = 1$. By the quotient rule, after substitution by our inequality and differential equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{u}{v} = \frac{u'v - v'u}{v^2} = \frac{u'v - \alpha uv}{v^2} \le \frac{\alpha uv - \alpha uv}{v^2} = 0 \tag{3.29}$$

so u/v as a function is decreasing, and must be less than its initial value,

$$\frac{u(t)}{v(t)} \le \frac{u(a)}{v(a)} = u(a) \tag{3.30}$$

which when reorganized yields

$$u(t) \le u(a)v(t) = u(a)\exp\biggl(\int_a^t \alpha(s)\,\mathrm{d}s\biggr) \tag{3.31}$$

as desired. □

Observe that Grönwall's lemma applies for only linear differential inequalities. We shall now show another Grönwall-type inequality regarding nonlinear differential inequalities, proven by [Perov, 1959]. A reference to this is provided in [Mitrinović et al., 1991] Chapter XII, Theorem 9.1.

Theorem 3.9. *Suppose $I = \{t : t \ge a\} \subseteq \mathbb{R}$ denotes an interval. Let $\alpha, \beta \in C^0(I)$ be non-negative continuous functions, and u be a non-negative function, all defined on I. If $u(t)$ satisfies the integral inequality*

$$u(t) \le u_0 + \int_a^t \alpha(s)u(s) + \beta(s)u^p(s)\,\mathrm{d}s \tag{3.32}$$

where $u_0 \ge 0$ and $p \in [0, 1]$, then $u(t)$ is bounded by the following.

$$u(t) \le \biggl[u_0^{1-p}\exp\biggl((1-p)\int_a^t \alpha(s)\,\mathrm{d}s\biggr) + (1-p)\int_a^t \beta(s)\exp\biggl((1-p)\int_s^t \alpha(r)\,\mathrm{d}r\biggr)\,\mathrm{d}s\biggr]^{1/(1-p)} \tag{3.33}$$

Proof. Let $v(t)$ be the solution to the corresponding integral equation.

$$v(t) = u_0 + \int_a^t \alpha(s)v(s) + \beta(s)v^p(s)\,\mathrm{d}s \tag{3.34}$$

There is a corresponding differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \alpha(t)v(t) + \beta(t)v^p(t) \tag{3.35}$$

where $v(a) = u_0$ is our initial condition.

To solve this differential equation, we first make a substitution $w = v^{1-p}$, which yields

$$\frac{\mathrm{d}w}{\mathrm{d}t} = (1-p)v^{-p}\frac{\mathrm{d}v}{\mathrm{d}t} = (1-p)v^{-p}(\alpha v + \beta v^p) = (1-p)(\alpha v^{1-p} + \beta) = (1-p)(\alpha w + \beta) \tag{3.36}$$

with the initial condition $w(a) = v^{1-p}(a) = u_0^{1-p}$. Observe that this new differential equation is linear, and has a well-known solution.

$$
\begin{aligned}
w(t) &= \exp\left((1-p)\int_a^t \alpha(s)\,\mathrm{d}s\right)\left[u_0^{1-p} + \int_a^t (1-p)\beta(s)\exp\left(-(1-p)\int_a^s \alpha(r)\,\mathrm{d}r\right)\mathrm{d}s\right] \\
&= u_0^{1-p}\exp\left((1-p)\int_a^t \alpha(s)\,\mathrm{d}s\right) + \int_a^t (1-p)\beta(s)\exp\left(-(1-p)\int_s^t \alpha(r)\,\mathrm{d}r\right)\mathrm{d}s \qquad (3.37)
\end{aligned}
$$

Since $v = w^{1/(1-p)}$, $v(t)$ is exactly the right side of Equation 3.33. But by construction, $u(t) \leq v(t)$, and we have our desired inequality. □

We remark that a special case of this theorem allows us to get a different bound on all first order linear differential inequalities. It is a slightly looser bound, but it allows us to perform less computation.

Corollary 3.10. *Suppose $I = \{t : t \geq a\} \subseteq \mathbb{R}$ denotes an interval. Let $\alpha, \beta \in C^0(I)$ be non-negative continuous functions, and u a non-negative function, all defined on I. If u satisfies the integral inequality*

$$
u(t) \leq u_0 + \int_a^t \alpha(s)u(s) + \beta(s)\,\mathrm{d}s \qquad (3.38)
$$

where $u_0 \geq 0$, then $u(t)$ is bounded by the following solution.

$$
u(t) \leq \int_a^t \beta(s)\,\mathrm{d}s + u_0\exp\left(\int_a^t \alpha(s)\,\mathrm{d}s\right) \qquad (3.39)
$$

Proof. From the theorem we already have

$$
u(t) \leq \int_a^t \beta(s)\exp\left(-\int_s^t \alpha(r)\,\mathrm{d}r\right)\mathrm{d}s + u_0\exp\left(\int_a^t \alpha(s)\,\mathrm{d}s\right) \qquad (3.40)
$$

But since α is nonnegative, the exponent is less than 1, which proves our corollary. □

Chapter 4

Physical Problem and Results

Throughout this chapter, we shall adopt a few conventions that are often used throughout existing literature, where a hatted quantity $\hat{a}$ denotes a quantity with dimension, a denotes its dimensionless counterpart; and bolded quantites $\mathbf{a}$ are either vectors or tensors.

4.1 Governing Equations

Consider two miscible fluids A and B with densities $\hat{\rho}_A > \hat{\rho}_B$ respectively, with fluid A placed on top of fluid B. Assuming incompressibility, the Navier-Stokes equations for this system can be written as

$$\hat{\rho}\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} = -\boldsymbol{\nabla}\hat{p} + \hat{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{\tau}} + \hat{\rho}\hat{\mathbf{g}} \tag{4.1}$$

$$\hat{\boldsymbol{\nabla}} \cdot \hat{\mathbf{u}} = 0 \tag{4.2}$$

$$\frac{\partial c}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \hat{\boldsymbol{\nabla}} c = \hat{D}\hat{\nabla}^2 c \tag{4.3}$$

where $\hat{\mathbf{u}} = (\hat{u}, \hat{v}, \hat{w})$ is the flow velocity, $\hat{\mathbf{g}} = (0, 0, -\hat{g})$ is the gravitational acceleration, c is the concentration of fluid A. We shall consider the case where the diffusion coefficient $\hat{D}$ is constant and the viscosity μ is a function of concentration c and shear rate $\hat{\dot{\boldsymbol{\gamma}}}$. The advection term is omitted: under the assumption that the velocity is small, advection amounts to being a quadratically small term. We also assume the Boussinesq approximation, where the volume of the fluid is unaffected by the concentration changes, but the density is. The density ρ shall be a linear function of concentration c, written as

$$\hat{\rho} = \hat{\rho}_B + c(\hat{\rho}_A - \hat{\rho}_B) \tag{4.4}$$

where at $c = 0$, the fluid is entirely fluid B, and at $c = 1$, the fluid is entirely fluid A.

We can now nondimensionalize the problem using the following scales, where $\hat{L}$ is a characteristic length, and $\hat{\mu}_0$ is some representative viscosity value:

$$L^* = \hat{L} \qquad \text{length scale} \tag{4.5}$$

$$u^* = \frac{\Delta\hat{\rho}\hat{g}\hat{L}^2}{\hat{\mu}_0} \qquad \text{velocity scale} \tag{4.6}$$

$$t^* = \frac{\hat{\mu}_0}{\Delta\hat{\rho}\hat{g}\hat{L}} \qquad \text{time scale} \tag{4.7}$$

$$p^* = \Delta\hat{\rho}\hat{g}\hat{L} \qquad \text{pressure scale} \tag{4.8}$$

$$\varepsilon = \frac{\Delta\hat{\rho}}{\hat{\rho}_B} \qquad \text{density ratio.} \tag{4.9}$$

Suppose our concentration has a leading term that only varies over height:

$$c = c_0(z) + \varepsilon c_1(x, y, z, t). \tag{4.10}$$

Then from Equation 4.4, the density must be

$$\rho = \rho_B + c_0\Delta\rho + \varepsilon c_1 \Delta\rho, \tag{4.11}$$

and pressure should have the form

$$p = \bar{p} + \rho g \Delta z = \bar{p} + \int (\rho_B + c_0\Delta\rho) g \,\mathrm{d}z + \varepsilon p_1 = \bar{p} + p_0(z) + \varepsilon p_1(x, y, z, t), \tag{4.12}$$

where the integral is taken from some reference point, assumed to be the origin of our coordinate system, to any arbitrary height.

We shall treat viscosity as a function of concentration, but assume that the perturbative term c_1 is not sufficient to cause a major disturbance in viscosity. Hence viscosity is a function of c_0, written as

$$\mu = \mu_0(c_0). \tag{4.13}$$

To extend our analysis to non-Newtonian fluids, we assume our fluids to be the simplest yield stress fluid, wbich is modeled by the Bingham model [Bingham, 1922]. Below some yield stress value τ_Y, we assume that the fluid is static, and only when the yield stress is exceeded, the fluid exhibits viscous behaviour as governed by Equation 4.13. Mathematically, we can write it as the piecewise function

$$\begin{cases} \dot{\boldsymbol{\gamma}} = \mathbf{0} & \tau \leq \tau_Y \\ \boldsymbol{\tau} = \left(\mu_0 + \frac{\tau_Y}{\dot{\gamma}}\right)\dot{\boldsymbol{\gamma}} & \tau > \tau_Y. \end{cases} \tag{4.14}$$

Of course, since the yield stress of fluids A and B might be different, yield stress is treated as a function of concentration $\tau_Y(c_0)$, again assuming that the concentration perturbations are not sufficient to cause a noticeable change in yield stress.

Here we alert the reader to the distinction between the tensors $\boldsymbol{\tau}, \dot{\boldsymbol{\gamma}}$ and their corresponding scalars $\tau, \dot{\gamma}$, which obey the following relationship:

$$\tau = \sqrt{\frac{1}{2}\boldsymbol{\tau} : \boldsymbol{\tau}} \qquad \dot{\gamma} = \sqrt{\frac{1}{2}\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}}. \tag{4.15}$$

We can additionally assume that initially our fluid is undisturbed and stable, so the velocity field must be

$$\mathbf{u} = \varepsilon \mathbf{u}_1(x, y, z, t) = \varepsilon \begin{bmatrix} u \\ v \\ w \end{bmatrix}. \tag{4.16}$$

After balancing the leading terms from $\boldsymbol{\nabla} p$ and $\rho\mathbf{g}$, and keeping terms of order ε, this yields the following set of dimensionless Navier-Stokes equations:

$$\mathrm{Re}\,\frac{\partial \mathbf{u}_1}{\partial t} = -\boldsymbol{\nabla} p_1 + \mu_0 \boldsymbol{\nabla}\cdot\dot{\boldsymbol{\gamma}} + \tau_Y \boldsymbol{\nabla}\cdot\frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}} + \left(\mu_0' + \frac{\tau_Y'}{\dot{\gamma}}\right)\frac{\mathrm{d}c_0}{\mathrm{d}z}\begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} - c_1 \boldsymbol{\nabla} z \tag{4.17}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}_1 = 0 \tag{4.18}$$

$$\frac{\partial c_1}{\partial t} + w\frac{\mathrm{d}c_0}{\mathrm{d}z} = \frac{1}{\mathrm{Pe}}\nabla^2 c_1 \tag{4.19}$$

where the Péclet and Reynolds numbers are

$$\mathrm{Pe} = \frac{\Delta\hat{\rho}\hat{g}\hat{L}^3}{\hat{\mu}_0\hat{D}} \qquad \mathrm{Re} = \frac{\hat{\rho}_B\Delta\hat{\rho}\hat{g}\hat{L}^3}{\hat{\mu}_0^2}. \tag{4.20}$$

Since we have chosen a singular length scale L^*, let us solve this problem in a domain Ω, representing a prismatic pipe with a base that has characteristic length L^*. At the walls of the pipe, we apply a no-slip boundary condition, which implies that at the wall the fluid adheres to the wall perfectly, and

$$\mathbf{u}|_{\partial\Omega} = 0 \qquad \text{except at top and bottom of pipe.} \tag{4.21}$$

Lastly we assume that all our perturbations are vertically periodic, with period being our characteristic length L^*. This allows the perturbations to take the form of multiples of $\sin(kz)$ or $\cos(kz)$, for $k \in \mathbb{Z}_{>0}$.

4.2 Energy Considerations

In this section, we analyze the linear stability of our system via the construction of upper bounds on our energies. As discussed in Section 2.3.2, a system is considered stable if any perturbation eventually decays to zero, or at least gets arbitrarily close to zero.

4.2.1 Energy Equations

Let us now take the inner product of Equation 4.17 with $\mathbf{u}$, and then integrate over the entire domain Ω. This amounts to constructing a term corresponding to kinetic energy on the left side, and yields an equation as shown below, obtained using integration by parts and Stokes' theorem:

$$\begin{aligned}\frac{1}{2}\,\mathrm{Re}\int_\Omega \frac{\partial \mathbf{u}_1^2}{\partial t}\,\mathrm{d}V = &-\int_{\Lambda_B}^{\Lambda_T} p_1 w\,\mathrm{d}A + \int_{\Lambda_B}^{\Lambda_T}\left(\mu_0 + \frac{\tau_Y}{\dot\gamma}\right)\mathbf{u}_1 \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z}\end{bmatrix}\mathrm{d}A \\ &- \int_\Omega \mu_0 \dot\gamma^2\,\mathrm{d}V - \int_\Omega \tau_Y \dot\gamma\,\mathrm{d}V - \int_\Omega c_1 w\,\mathrm{d}V\,.\end{aligned} \tag{4.22}$$

Note that the integral with Λ_T and Λ_B denotes a difference between two integrals:

$$\int_{\Lambda_B}^{\Lambda_T} a\,\mathrm{d}A = \int_{\Lambda_T} a\,\mathrm{d}A - \int_{\Lambda_B} a\,\mathrm{d}A\,. \tag{4.23}$$

The first two terms are quantities that describe the differences between the top and bottom of our domain, so by periodicity of our domain in the z direction, those two terms vanish.

Let our kinetic energy be $\mathcal{K} = \frac{1}{2}\left\langle \mathbf{u}_1^2 \right\rangle$. Then our kinetic energy equation is

$$\mathrm{Re}\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = -\left\langle \mu_0 \dot\gamma^2 \right\rangle - \left\langle \tau_Y \dot\gamma \right\rangle - \left\langle c_1 w \right\rangle, \tag{4.24}$$

where the expectation value $\langle a \rangle$ is the integral over the domain Ω, which represents the average value of a over said domain.

We can perform a similar procedure on Equation 4.19, by multiplying with c_1 and then integrating over Ω. This time we are constructing a term corresponding to the potential energy of the perturbation on the left side, and yields a second equation

$$\mathrm{Pe}\,\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = -\,\mathrm{Pe}\left\langle \frac{\mathrm{d}c_0}{\mathrm{d}z} c_1 w \right\rangle - \left\langle |\boldsymbol{\nabla} c_1|^2 \right\rangle, \tag{4.25}$$

if we let the potential energy be $\mathcal{U} = \frac{1}{2}\left\langle c_1^2 \right\rangle$.

Furthermore, if we let the total energy of the system be $\mathcal{E} = \mathrm{Re}\,\mathcal{K} + \mathrm{Pe}\,\mathcal{U}$, then by combining Equations 4.24 & 4.25, we get the following total energy equation:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\langle c_1 w\rangle - \mathrm{Pe}\left\langle \frac{\mathrm{d}c_0}{\mathrm{d}z} c_1 w \right\rangle - \left\langle \mu_0 \dot{\boldsymbol{\gamma}}^2 \right\rangle - \langle \tau_Y \dot{\gamma}\rangle - \left\langle |\boldsymbol{\nabla} c_1|^2 \right\rangle. \tag{4.26}$$

4.2.2 Upper Bounds

Before we can analyze the stability of the system using the energy equations, we must bound each of the five terms with quantities that are more familiar and easier to work with.

There are a variety of ways to bound the first term. One way is to first apply Hölder's inequality:

$$-\langle c_1 w\rangle \leq \left(\sup_{\Omega,t} |c_1|\right) \|w\|_1. \tag{4.27}$$

We know that c_1 is bounded inside Ω, since by the triangle inequality we have

$$|c_1| = \frac{1}{\varepsilon}|c - c_0| \leq \frac{1}{\varepsilon}(|c| + |c_0|) \leq \frac{2}{\varepsilon}, \tag{4.28}$$

where both $|c|$ and $|c_0|$ are bounded above by 1, by the definition of concentration. Therefore we can conclude that $|c_1| \leq c_{\max}$ is bounded by some constant $c_{\max}$ that only depends on Ω and ε. Hence our first term can be bounded by

$$-\langle c_1 w\rangle \leq c_{\max}\|w\|_1 \leq c_{\max}\|\mathbf{u}_1\|_1. \tag{4.29}$$

This can be further relaxed by an application of Corollary 3.4, which yields $C\|\mathbf{u}_1\|_{3/2} \geq \|\mathbf{u}_1\|_1$ for some C dependent only on Ω, which we shall lump into $c_{\max}$ to get

$$-\langle c_1 w\rangle \leq c_{\max}\|\mathbf{u}_1\|_{3/2}. \tag{4.30}$$

Another way to bound the first term is to use the Cauchy-Schwarz inequality, and we apply the bound on the product of norms (Theorem 3.2), yielding

$$-\langle c_1 w\rangle \leq \|c_1\|_2\|w\|_2 \leq \|c_1\|_2\|\mathbf{u}_1\|_2 \leq \frac{1}{2\sqrt{\mathrm{Re}\,\mathrm{Pe}}}\left(\mathrm{Pe}\,\|c_1\|_2^2 + \mathrm{Re}\,\|\mathbf{u}_1\|_2^2\right) = \frac{1}{\sqrt{\mathrm{Re}\,\mathrm{Pe}}}\mathcal{E}. \tag{4.31}$$

Observe that the second term can be bound using the exact same inequalities as the first term, so we simply state the following results:

$$-\left\langle \frac{\mathrm{d}c_0}{\mathrm{d}z} c_1 w \right\rangle \leq G\langle c_1 w\rangle \leq c_{\max} G\|\mathbf{u}_1\|_1 \tag{4.32}$$

$$-\left\langle \frac{\mathrm{d}c_0}{\mathrm{d}z} c_1 w \right\rangle \leq G\langle c_1 w\rangle \leq G\|c_1\|_2\|\mathbf{u}_1\|_2 \leq \frac{G}{\sqrt{\mathrm{Re}\,\mathrm{Pe}}}\mathcal{E}. \tag{4.33}$$

We define $G = |\sup_\Omega \mathrm{d}c_0/\mathrm{d}z|$ to be the maximum concentration gradient, with the supremum taken over the entire domain Ω. We argue that G is finite because it is unreasonable to consider the case where the boundary between the two miscible fluids is sharp and horizontal.

As in Theorem 1 of [Zhang et al., 2006], by applying Poincaré's inequality on the third term, we have

$$\left\langle \mu_0 \dot{\boldsymbol{\gamma}}^2 \right\rangle \geq \mu_{\min}\left\langle \dot{\boldsymbol{\gamma}}^2 \right\rangle = \mu_{\min}\left\langle |\boldsymbol{\nabla}\mathbf{u}_1|^2 \right\rangle \geq C_u \mu_{\min}\left\langle |\mathbf{u}_1|^2 \right\rangle = C_u\mu_{\min}\left\langle \mathbf{u}_1^2 \right\rangle, \tag{4.34}$$

where $\mu_{\min} = \min_{0\le c\le 1} \mu_0$ is the minimum viscosity and C_u is a constant dependent only on Ω.

For the fourth term, [Témam and Strang, 1980] shows that it can be bounded as

$$\langle \tau_Y \dot{\gamma} \rangle \ge \tau_{\min} \langle \dot{\gamma} \rangle \ge C_\gamma \tau_{\min} \|\mathbf{u}_1\|_{3/2} \tag{4.35}$$

where $\tau_{\min} = \min_{0\le c\le 1} \tau_Y$ is the minimum yield stress and C_γ is a constant depending only on Ω. But the same application of Corollary 3.4 as above yields $C\|\mathbf{u}_1\|_{3/2} \ge \|\mathbf{u}_1\|_1$ for some C dependent again only on Ω, which we shall lump into C_γ to get

$$\langle \tau_Y \dot{\gamma} \rangle \ge C_\gamma \tau_{\min} \|\mathbf{u}_1\|_1. \tag{4.36}$$

On the last term we can apply Poincaré's inequality to get

$$\left\langle |\boldsymbol{\nabla} c_1|^2 \right\rangle \ge C_c \left\langle |c_1|^2 \right\rangle = C_c \left\langle c_1^2 \right\rangle, \tag{4.37}$$

where C_c is a constant dependent only on Ω.

4.2.3 Newtonian Fluids

For now, suppose that both of our fluids are Newtonian. Then by definition yield stress $\tau_Y = 0$, giving us the following set of energy equations, which follows directly from Equations 4.24-4.26:

$$\mathrm{Re}\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = -\left\langle \mu_0 \dot{\gamma}^2 \right\rangle - \langle c_1 w \rangle \tag{4.38}$$

$$\mathrm{Pe}\,\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = -\,\mathrm{Pe}\left\langle \frac{\mathrm{d}c_0}{\mathrm{d}z} c_1 w \right\rangle - \left\langle |\boldsymbol{\nabla} c_1|^2 \right\rangle \tag{4.39}$$

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\langle c_1 w \rangle - \mathrm{Pe}\left\langle \frac{\mathrm{d}c_0}{\mathrm{d}z} c_1 w \right\rangle - \left\langle \mu_0 \dot{\gamma}^2 \right\rangle - \left\langle |\boldsymbol{\nabla} c_1|^2 \right\rangle. \tag{4.40}$$

We can immediately apply bounds from Equations 4.31, 4.33, 4.34, & 4.37 and get an inequality on the time evolution of the total energy, which is

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \le \frac{1 + \mathrm{Pe}\,G}{\sqrt{\mathrm{Re}\,\mathrm{Pe}}}\mathcal{E} - 2C_u\mu_{\min}\mathcal{K} - 2C_c\mathcal{U} \le \frac{1 + \mathrm{Pe}\,G}{\sqrt{\mathrm{Re}\,\mathrm{Pe}}}\mathcal{E} - 2\min\{C_u\mu_{\min}, C_c\}\mathcal{E}. \tag{4.41}$$

By Grönwall's lemma, the energy is bounded as

$$\mathcal{E}(t) \le \mathcal{E}(0)\exp\left(\left(\frac{1 + \mathrm{Pe}\,G}{\sqrt{\mathrm{Re}\,\mathrm{Pe}}} - \alpha\right)t\right), \qquad \text{where } \alpha = 2\min\{C_u\mu_{\min}, C_c\}, \tag{4.42}$$

which under the assumption that if

$$\alpha > \frac{1 + \mathrm{Pe}\,G}{\sqrt{\mathrm{Re}\,\mathrm{Pe}}}, \tag{4.43}$$

then our total energy is bounded by a decaying exponential, which implies our system is stable.

Since kinetic and potential energy are both, by definition, less than total energy, we can consequently conclude that both are going to decay exponentially, and as a result, $\|\mathbf{u}_1\|_2$ and $\|c_1\|_2$ will also decay exponentially.

4.2.4 Non-Newtonian Fluids

Let us restate Equations 4.24-4.26, the set of energy equations, in their full form, with the yield stress term present:

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -\left\langle c_1 w\right\rangle - \mathrm{Pe}\left\langle \frac{\mathrm{d}c_0}{\mathrm{d}z} c_1 w\right\rangle - \left\langle \mu_0 \dot{\boldsymbol{\gamma}}^2\right\rangle - \left\langle \tau_Y \dot{\gamma}\right\rangle - \left\langle |\boldsymbol{\nabla} c_1|^2\right\rangle \tag{4.44}$$

$$\mathrm{Re}\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = -\left\langle \mu_0 \dot{\gamma}^2\right\rangle - \left\langle \tau_Y \dot{\gamma}\right\rangle - \left\langle c_1 w\right\rangle \tag{4.45}$$

$$\mathrm{Pe}\,\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = -\,\mathrm{Pe}\left\langle \frac{\mathrm{d}c_0}{\mathrm{d}z} c_1 w\right\rangle - \left\langle |\boldsymbol{\nabla} c_1|^2\right\rangle. \tag{4.46}$$

We first notice that the fourth term $-\left\langle \tau_Y \dot{\gamma}\right\rangle$ is entirely negative, so the analysis from Section 4.2.3 still applies. However, we wish to find a tighter bound, since energy should intuitively dissipate quicker with extra yield stress.

We use Equations 4.30, 4.32, 4.34, 4.35 & 4.37, which yields

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \le -C_u \mu_{\min}\|\mathbf{u}_1\|^2 - C_c\|c_1\|^2 - (C_\gamma \tau_{\min} - c_{\max}(1 + \mathrm{Pe}\,G))\|\mathbf{u}_1\|_{3/2} \tag{4.47}$$

$$\mathrm{Re}\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} \le -C_u \mu_{\min}\|\mathbf{u}_1\|^2 - (C_\gamma \tau_{\min} - c_{\max})\|\mathbf{u}_1\|_{3/2} \tag{4.48}$$

$$\mathrm{Pe}\,\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} \le c_{\max}\,\mathrm{Pe}\,G\|\mathbf{u}_1\|_{3/2} - C_c\|c_1\|^2. \tag{4.49}$$

We make the assumption that the $L^{3/2}$ norm is equivalent to the L^2 norm for all $\mathbf{u}_1$ in the function space of solutions. Then we can immediately write down

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \le -C_u \mu_{\min}\|\mathbf{u}_1\|^2 - C_c\|c_1\|^2 - (C_\gamma \tau_{\min} - c_{\max}(1 + \mathrm{Pe}\,G))\|\mathbf{u}_1\| \tag{4.50}$$

$$\mathrm{Re}\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} \le -C_u \mu_{\min}\|\mathbf{u}_1\|^2 - (C_\gamma \tau_{\min} - c_{\max})\|\mathbf{u}_1\| \tag{4.51}$$

$$\mathrm{Pe}\,\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} \le c_{\max}\,\mathrm{Pe}\,G\|\mathbf{u}_1\| - C_c\|c_1\|^2. \tag{4.52}$$

Dividing the kinetic energy inequality by $\|\mathbf{u}_1\|$, this becomes a velocity inequality,

$$\mathrm{Re}\,\frac{\mathrm{d}\|\mathbf{u}_1\|}{\mathrm{d}t} \le -C_u \mu_{\min}\|\mathbf{u}_1\| - (C_\gamma \tau_{\min} - c_{\max}). \tag{4.53}$$

If $C_\gamma \tau \mathrm{min} \ge c_{\max}$, then by Corollary 3.10,

$$\|\mathbf{u}_1\|(t) \le -\frac{1}{\mathrm{Re}}(C_\gamma \tau_{\min} - c_{\max})t + \|\mathbf{u}_1\|(0)\exp\left(-\frac{C_u \mu_{\min}}{\mathrm{Re}}t\right), \qquad \text{for } t \le t_0,$$

for some $t_0 > 0$, since we know that at large t, the linear term dominates, and thus formally evaluating the right side will get us to some negative value, so before that happens, there must be some $\|\mathbf{u}_1\|(t_0) = 0$. Afterwards, $\|\mathbf{u}_1\|(t) = 0$ for $t > t_0$. This demonstrates that under our assumption, $\|\mathbf{u}_1\|$ decays to zero in finite time.

We produce a slightly better bound on $\|c_1\|$, since we know that $\|\mathbf{u}_1\|$ vanishes after t_0, and we should have a tighter bound when $t > t_0$. The potential energy inequality becomes

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} \le -\frac{1}{\mathrm{Pe}}\left\langle |\boldsymbol{\nabla} c_1|^2\right\rangle \le -\frac{C_c}{\mathrm{Pe}}\|c_1\|^2 \qquad \text{for } t > t_0. \tag{4.54}$$

Rewriting this in terms of $\|c_1\|$ yields

$$\frac{\mathrm{d}\|c_1\|}{\mathrm{d}t} \leq -\frac{C_c}{\mathrm{Pe}}\|c_1\| \qquad \text{for } t > t_0, \tag{4.55}$$

which by Grönwall's lemma gives

$$\|c_1\|(t) \leq \|c_1\|(t_0)e^{-C_c t/\mathrm{Pe}} \leq \sqrt{2\mathcal{E}(0)}\exp\left(\left(\frac{1+\mathrm{Pe}\,G}{2\sqrt{\mathrm{Re}}} - \frac{\alpha}{2} - \frac{C_c}{\mathrm{Pe}}\right)t\right) \qquad \text{for } t > t_0. \tag{4.56}$$

4.2.5 Non-Newtonian Fluids, Differently

Alternatively, we can bound the terms using Equations 4.29, 4.32, 4.34, 4.36 & 4.37, which yields

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \leq -C_u\mu_{\min}\|\mathbf{u}_1\|^2 - C_c\|c_1\|^2 - (C_\gamma\tau_{\min} - c_{\max}(1+\mathrm{Pe}\,G))\|\mathbf{u}_1\|_1 \tag{4.57}$$

$$\mathrm{Re}\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} \leq -C_u\mu_{\min}\|\mathbf{u}_1\|^2 - (C_\gamma\tau_{\min} - c_{\max})\|\mathbf{u}_1\|_1 \tag{4.58}$$

$$\mathrm{Pe}\,\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} \leq c_{\max}\,\mathrm{Pe}\,G\|\mathbf{u}_1\|_1 - C_c\|c_1\|^2. \tag{4.59}$$

An application of the Gagliardo-Nirenberg inequality, with $j=0, m=1, n=3, p=2, q=1, r=2, \theta=3/5$ gives us the bound

$$\|\mathbf{u}_1\|_1 \geq C_{GN}\frac{\|\mathbf{u}_1\|_2^{5/2}}{\|\mathbf{u}_1\|_{1,2}^{3/2}} \tag{4.60}$$

for some constant C_{GN} dependent on Ω.

We make the critical, but possibly naïve assumption here that $\|\boldsymbol{\nabla}\mathbf{u}_1\|/\|\mathbf{u}_1\|$ is bounded above for all time, and that bound is not dependent on $\mathbf{u}_1$ itself. This will let us get a lower bound on this ratio of norms,

$$\left(\frac{\|\mathbf{u}_1\|_2}{\|\mathbf{u}_1\|_{1,2}}\right)^{3/2} = \left(\frac{\|\mathbf{u}_1\|}{\|\mathbf{u}_1\| + \|\boldsymbol{\nabla}\mathbf{u}_1\|}\right)^{3/2} \geq \left(\frac{\|\mathbf{u}_1\|}{\|\mathbf{u}_1\| + \sup_t\|\boldsymbol{\nabla}\mathbf{u}_1\|}\right)^{3/2} = u_{\text{crit}}, \tag{4.61}$$

and we shall term u_{crit} the critical velocity ratio.

Then the kinetic energy inequality can be written as

$$\mathrm{Re}\,\frac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} \leq -C_u\mu_{\min}\|\mathbf{u}_1\|^2 - C_{GN}u_{\text{crit}}(C_\gamma\tau_{\min} - c_{\max})\|\mathbf{u}_1\|. \tag{4.62}$$

When we divide by $\|\mathbf{u}_1\|$, this becomes a velocity inequality

$$\mathrm{Re}\,\frac{\mathrm{d}\|\mathbf{u}_1\|}{\mathrm{d}t} \leq -C_u\mu_{\min}\|\mathbf{u}_1\| - C_{GN}u_{\text{crit}}(C_\gamma\tau_{\min} - c_{\max}).$$

If $C_\gamma\tau\min \geq c_{\max}$, then by Corollary 3.10,

$$\|\mathbf{u}_1\|(t) \leq -\frac{C_{GN}u_{\text{crit}}}{\mathrm{Re}}(C_\gamma\tau_{\min} - c_{\max})t + \|\mathbf{u}_1\|(0)\exp\left(-\frac{C_u\mu_{\min}}{\mathrm{Re}}t\right) \qquad \text{for } t \leq t_0,$$

for some $t_0 > 0$, and afterwards, $\|\mathbf{u}_1\|(t) = 0$ for $t > t_0$, by the exact same logic as the previous section. We have demonstrated that under a different assumption, $\|\mathbf{u}_1\|$ also decays to zero in finite time.

The calculation for $\|c_1\|$ after t_0 remains the same as the previous section.

4.3 Validity of Assumptions

In letting the flow field extinguish in finite time, we have made two assumptions that we need to justify. We shall in turn examine these two cases.

4.3.1 Equivalent Norms

In Section 4.2.4, we assumed that our solutions have equivalent $L^{3/2}$ and L^2 norms. We shall prove a theorem about L^p spaces before undertaking that discussion.

Theorem 4.1. *Suppose Ω is a domain, and $1 < p, q < \infty$. Let $U \subseteq L^p(\Omega) \cap L^q(\Omega)$ be a subspace of functions, with U being closed in both $L^p(\Omega)$ and $L^q(\Omega)$. Then the L^p and L^q norms are equivalent on U.*

Proof. Consider $(U, \|\cdot\|_p)$ and $(U, \|\cdot\|_q)$ as two Banach spaces. We note that they are complete simply because they are closed subspaces. By the closed graph theorem[4], the identity map $(U, \|\cdot\|_p) \to (U, \|\cdot\|_q)$ is bounded, and the argument with p and q reversed gives us the other inequality. □

We shall prove a theorem (and a corollary) regarding functions of bounded norms that allows us to construct the function space of solutions to our Navier-Stokes equations.

Theorem 4.2. *Suppose Ω is a metric space, and $1 \leq p, q < \infty$. Consider the closed ball $B = \{f \in L^p(\Omega) : \|f\|_p \leq M\}$ for some finite $M > 0$. Then $B \cap L^q(\Omega)$ is a closed subset of $L^q(\Omega)$.*[5]

Proof. Consider any limit point f of $B \cap L^q(\Omega)$. Then there exists a sequence $\{f_n\} \subseteq B$ such that the sequence converges in the L^q norm, $\lim_{n\to\infty} \|f_n - f\|_q$. But convergence in the L^q norm implies there exists a subsequence $\{f_{n_k}\}$ that converges to f almost everywhere. Then by Fatou's lemma[6],

$$\|f\|_p^p = \int_\Omega |f|^p \,\mathrm{d}\mu \leq \liminf \int_\Omega |f_{n_k}|^p = \liminf \|f_{n_k}\|_p^p \leq M^q$$

Hence the limit point $f \in B$, and B is closed. □

Corollary 4.3. *Suppose Ω is a finite measure space, and $1 \leq p < q < \infty$. Consider the closed ball $B = \{f \in L^q(\Omega) : \|f\|_q \leq M\}$ for some $M > 0$. Then B is a closed subset of $L^p(\Omega)$.*

Proof. For finite measure spaces, we have the embedding $L^q(\Omega) \subseteq L^p(\Omega)$. Hence $B \cap L^p(\Omega) = B$, and we can directly apply Theorem 4.2. □

Consider the space of all solutions $\mathbf{u}_1$ to our (linearized) Navier-Stokes equation, with $\|\mathbf{u}_1\|_2 \leq 2\sqrt{\mathcal{E}_{\max}}$ for some finite energy $\mathcal{E}_{\max}$, and let that set of functions be U. It is immediate that U is a subspace of a closed L^2 ball, which by Corollary 4.3, gives us that said ball is also closed in $L^{3/2}$; this allows us to say, by Theorem 4.1, that the closed ball, and hence U, has equivalent $L^{3/2}$ and L^2 norms as desired.

We remark that the equivalence of L^p norms on bounded functions have consequences beyond the analysis of viscosity-driven linear stability problems. For example, finite time decay of the velocity can be guaranteed in three (or even more) dimensions; and for any concentration gradients, provided that every single term in the energy equation is bounded above by some $\|f\|_p$, for which the corresponding $\|f\|_2$ is bounded by the energy.

[4] The closed graph theorem states that for any two Banach spaces X, Y and a closed linear map $T : X \to Y$, T is a bounded linear operator. A reference is given in [Folland, 2013], Theorem 5.12.

[5] We extend a token of gratitude to Oakley and Christian for the discussion that culminated in this theorem.

[6] Fatou's lemma states that for any positive sequence of functions $\{f_n\}$, then $\int \liminf f_n \leq \liminf \int f_n$. A reference in given in [Folland, 2013], Theorem 2.18.

4.3.2 Caccioppoli's Inequality

However, if we consider our second approach, to write down Equation 4.61, we made the assumption that there exists some constant C dependent on Ω only, such that for any linear perturbation $\mathbf{u}_1$ to our Navier-Stokes equations,

$$\|\boldsymbol{\nabla}\mathbf{u}_1\| \leq C\|\mathbf{u}_1\|. \tag{4.63}$$

This is our desired criterion, called Caccioppoli's inequality; it does not hold in general for all $\mathbf{u}$ in any large class of integrable functions. We shall discuss in this section, the possibility of obtaining such an inequality for our system.

We shall admit that such an inequality has yet to be proven. Nevertheless, attempts have been made to prove similar inequalities on different sets of Navier-Stokes equations. In particular, [Jin, 2013] provided a local version of the Caccioppoli inequality for Newtonian fluids, while [Jin and Kang, 2017] proved a similar inequality in the case of non-Newtonian fluids. However, [Chang and Kang, 2018] has shown that such a class of inequalities do not hold near boundary for Newtonian fluids; and in general, these inequalities do not work directly on idealized models with yield stress, such as the Bingham model.

4.4 Interpretation of Results

We remind the reader that the analysis done in this chapter only holds for a sufficiently small perturbation. For a larger perturbation, the time required to reach zero velocity increases. Eventually, for a large enough perturbation, the time scale will reach a point where the diffusive and inertial effects cannot be ignored. These scales will be determined by the Reynolds and Péclet numbers.

Nevertheless, our result on finite-time decay of velocity allows us to exert finer control on yield-stress fluids, potentially allowing us to develop technology that exploit this exact property. Furthermore, we hope that the validity of our assumption, and in particular the equivalence of norms demonstrated in Corollary 4.3, allows generalization of the finite-time decay phenomenon to other types of fluid gradients.

Chapter 5

Conclusion

In this thesis, we derived a set of Navier-Stokes equations for a concentration gradient of two Newtonian or Bingham fluids in a pipe. We considered the kinetic and potential energies of the system, and its evolution through time. We then applied upper bounds to terms in the energy equations, allowing us to determine two mathematical criteria where the velocity of the system decays to zero in finite time. Of these two criteria, the first one, the equivalence of L^p norms for velocity fields, is readily proven to be true for bounded initial energy; the second one, Caccioppoli's inequality, has not been proven to be true or false in this work.

Our results, in particular the equivalence of norms, potentially serve to solve a much larger class of problems regarding the energy stability of fluid gradients. While previous work indicates that the finite-time decay results in two dimensions cannot be easily extended into three dimensions, we have shown that, in fact, there exist physical and mathematical conditions in which finite-time decay can be immediately observed in three dimensions.

Ultimately, future work that remains to be done include an affirmation or a refutation of the second approach to finite-time decay, via Caccioppoli's inequality. There remain mathematically interesting details in proving or disproving this inequality for different types of non-Newtonian fluids. Another potential extension of this work would be to generalize the results of this thesis, both to other models of yield stress fluids, and to other types of fluid gradients.

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Appendix A

Derivations of Certain Equations

A.1 Equation 4.17

We start from Equation 4.1. On the left side of that equation, we substitute ρ with Equation 4.11, $\hat{\mathbf{u}} = \varepsilon u^* \mathbf{u}_1$ from Equations 4.6 & 4.16, and $\hat{t} = t^* t$ from Equation 4.7, getting

$$\hat{\rho}\frac{\partial \hat{\mathbf{u}}}{\partial t} = \left(\varepsilon(\hat{\rho}_B + c_0\Delta\hat{\rho}) + \varepsilon^2 c_1 \Delta\hat{\rho}\right)\frac{\Delta\hat{\rho}\hat{g}\hat{L}^2}{\hat{\mu}_0}\frac{\Delta\hat{\rho}\hat{g}\hat{L}}{\hat{\mu}_0}\frac{\partial u}{\partial t} = \varepsilon\hat{\rho}_B\frac{(\Delta\hat{\rho})^2\hat{g}^2\hat{L}^3}{\hat{\mu}_0^2}\frac{\partial u}{\partial t}. \tag{A.1}$$

since we can ignore the quadratic ε^2 term, and under the assumption that $\Delta\hat{\rho} \ll \hat{\rho}_B$, the $c_0\Delta\hat{\rho}$ term vanishes.

We now deal with the second term on the right side. By definition of yield stress, we have $\hat{\boldsymbol{\tau}} = \hat{\mu}\hat{\dot{\boldsymbol{\gamma}}}$, so we substitute $\hat{\mu} = \hat{\mu}_0\mu_0$ from Equation 4.13. Shear rate has dimensions velocity over length, so by Equations 4.5 & 4.6, we nondimensionalize with $\Delta\hat{\rho}\hat{g}\hat{L}/\hat{\mu}_0$, getting

$$\hat{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{\tau}} = \hat{\mu}_0\frac{\Delta\hat{\rho}\hat{g}\hat{L}}{\hat{\mu}_0}\frac{1}{\hat{L}}\boldsymbol{\nabla} \cdot \left(\left(\mu_0 + \frac{\tau_Y}{\dot{\gamma}}\right)\dot{\boldsymbol{\gamma}}\right) = \Delta\hat{\rho}\hat{g}\boldsymbol{\nabla} \cdot \left(\left(\mu_0 + \frac{\tau_Y}{\dot{\gamma}}\right)\dot{\boldsymbol{\gamma}}\right). \tag{A.2}$$

We apply the product rule on the divergence, and exploit its linearity:

$$\boldsymbol{\nabla} \cdot \left(\left(\mu_0 + \frac{\tau_Y}{\dot{\gamma}}\right)\dot{\boldsymbol{\gamma}}\right) = \boldsymbol{\nabla}\mu_0 \cdot \dot{\boldsymbol{\gamma}} + \mu_0\boldsymbol{\nabla} \cdot \dot{\boldsymbol{\gamma}} + \boldsymbol{\nabla}\tau_Y \cdot \frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}} + \tau_Y\boldsymbol{\nabla} \cdot \frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}}. \tag{A.3}$$

The first term here evaluates to

$$\boldsymbol{\nabla}\mu_0 \cdot \dot{\boldsymbol{\gamma}} = \mu_0'\frac{\partial(c_0 + \varepsilon c_1)}{\partial z}\begin{bmatrix}\dot{\gamma}_{xz}\\ \dot{\gamma}_{yz}\\ \dot{\gamma}_{zz}\end{bmatrix} = \varepsilon\mu_0'\frac{\mathrm{d}c_0}{\mathrm{d}z}\begin{bmatrix}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\\ 2\frac{\partial w}{\partial z}\end{bmatrix}, \tag{A.4}$$

since μ_0 only depends on c_0, which in turn only depends on z, so the $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ components of the gradient vanish. Moreover, the c_1 term is of order ε^2, which is negligible to us. The second term shall be left as is.

For the third term, we mimic the process on the first term, yielding

$$\boldsymbol{\nabla}\tau_Y \cdot \frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}} = \varepsilon\frac{\tau_Y'}{\dot{\gamma}}\frac{\mathrm{d}c_0}{\mathrm{d}z}\begin{bmatrix}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\\ 2\frac{\partial w}{\partial z}\end{bmatrix}. \tag{A.5}$$

The fourth term shall be left as is.

We can lastly look at the first and third terms together on the right side of Equation 4.1. Using Equations 4.8 & 4.11, we obtain

$$-\hat{\boldsymbol{\nabla}}\hat{p} + \hat{\rho}\hat{\mathbf{g}} = -\frac{\Delta\hat{\rho}\hat{g}\hat{L}}{\hat{L}}\boldsymbol{\nabla}(p_0 + \varepsilon p_1) + (\hat{\rho}_B + c_0\Delta\hat{\rho} - \varepsilon c_1\Delta\hat{\rho})\hat{g}\boldsymbol{\nabla}z. \tag{A.6}$$

Notice all the previous equations in this section are of order ε, so it would be natural to assume that the two terms of order unity here must balance each other out, leaving us with

$$\Delta\hat{\rho}\hat{g}\boldsymbol{\nabla} p_0 = (\hat{\rho}_B + c_0\Delta\hat{\rho})\hat{g}\boldsymbol{\nabla} z \tag{A.7}$$

$$-\boldsymbol{\nabla}\hat{p} + \hat{\rho}\hat{\mathbf{g}} = \varepsilon\Delta\hat{\rho}\hat{g}(-\boldsymbol{\nabla} p_1 - c_1\boldsymbol{\nabla} z). \tag{A.8}$$

We now combine all the terms together to get

$$\frac{\hat{\rho}_B(\Delta\hat{\rho})^2\hat{g}^2\hat{L}^3}{\hat{\mu}_0^2}\frac{\partial \mathbf{u}_1}{\partial t} = \Delta\hat{\rho}\hat{g}\left(-\boldsymbol{\nabla} p_1 + \mu_0\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\gamma}} + \tau_Y\boldsymbol{\nabla}\cdot\frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}} + \left(\mu_0' + \frac{\tau_Y'}{\dot{\gamma}}\right)\frac{\mathrm{d}c_0}{\mathrm{d}z}\begin{bmatrix}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\\ 2\frac{\partial w}{\partial z}\end{bmatrix} - c_1\boldsymbol{\nabla} z\right), \tag{A.9}$$

which simplifies down to

$$\frac{\hat{\rho}_B\Delta\hat{\rho}\hat{g}\hat{L}^3}{\hat{\mu}_0^2}\frac{\partial \mathbf{u}_1}{\partial t} = -\boldsymbol{\nabla} p_1 + \mu_0\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\gamma}} + \tau_Y\boldsymbol{\nabla}\cdot\frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}} + \left(\mu_0' + \frac{\tau_Y'}{\dot{\gamma}}\right)\frac{\mathrm{d}c_0}{\mathrm{d}z}\begin{bmatrix}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\\ 2\frac{\partial w}{\partial z}\end{bmatrix} - c_1\boldsymbol{\nabla} z \tag{A.10}$$

as desired.

A.2 Equation 4.24

We start from Equation 4.17. On the left side, observe that we can apply the product rule to get

$$\frac{1}{2}\,\mathrm{Re}\int_\Omega \frac{\partial \mathbf{u}_1^2}{\partial t}\,\mathrm{d}V = \frac{1}{2}\,\mathrm{Re}\int_\Omega \mathbf{u}_1\cdot\frac{\partial \mathbf{u}_1}{\partial t} + \frac{\partial \mathbf{u}_1}{\partial t}\cdot\mathbf{u}_1\,\mathrm{d}V = \mathrm{Re}\int_\Omega \mathbf{u}_1\cdot\frac{\partial \mathbf{u}_1}{\partial t}\,\mathrm{d}V \tag{A.11}$$

as desired.

The first term on the right side of Equation 4.17 can be integrated by parts, and applying the divergence theorem, we get

$$-\int_\Omega \mathbf{u}_1\cdot\boldsymbol{\nabla} p_1\,\mathrm{d}V = \int_\Omega p_1\boldsymbol{\nabla}\cdot\mathbf{u}_1\,\mathrm{d}V - \int_\Omega \boldsymbol{\nabla}\cdot(p_1\mathbf{u}_1)\,\mathrm{d}V = -\int_{\partial\Omega} p_1\mathbf{u}_1\cdot\mathbf{n}\,\mathrm{d}A = -\int_{\Lambda_B}^{\Lambda_T} p_1 w\,\mathrm{d}A\,, \tag{A.12}$$

where $\mathbf{n}$ is the outward normal. Notice that the first term disappears, since $\mathbf{u}$ is divergence-free (Equation 4.18).

The second term on the right side of the equation can also be integrated by parts, using a different product rule identity:

$$\int_\Omega \mu_0\mathbf{u}_1\cdot(\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\gamma}})\,\mathrm{d}V = \int_{\partial\Omega}\mu_0(\mathbf{n}\otimes\mathbf{u}_1):\dot{\boldsymbol{\gamma}}\,\mathrm{d}A - \int_\Omega \boldsymbol{\nabla}(\mu_0\mathbf{u}_1):\dot{\boldsymbol{\gamma}}\,\mathrm{d}V\,. \tag{A.13}$$

On the boundary we have

$$\int_{\partial\Omega}\mu_0(\mathbf{n}\otimes\mathbf{u}_1):\dot{\boldsymbol{\gamma}}\,\mathrm{d}A = \int_{\Lambda_B}^{\Lambda_T}\mu_0(u\dot{\boldsymbol{\gamma}}_{xz} + v\dot{\boldsymbol{\gamma}}_{yz} + w\dot{\boldsymbol{\gamma}}_{zz})\,\mathrm{d}A = \int_{\Lambda_B}^{\Lambda_T}\mu_0\mathbf{u}_1\cdot\begin{bmatrix}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\\ 2\frac{\partial w}{\partial z}\end{bmatrix}\mathrm{d}A\,, \tag{A.14}$$

where Λ_T and Λ_B denote the top and bottom area of the pipe opening. Inside our domain we have

$$
\begin{aligned}
-\int_\Omega \boldsymbol{\nabla}(\mu_0 \mathbf{u}_1) : \dot{\boldsymbol{\gamma}} \, \mathrm{d}V &= -\int_\Omega (\boldsymbol{\nabla}\mu_0 \otimes \mathbf{u}_1 + \mu_0 \boldsymbol{\nabla}\mathbf{u}_1) : \dot{\boldsymbol{\gamma}} \, \mathrm{d}V \\
&= -\int_\Omega \mu_0' \frac{\mathrm{d}c_0}{\mathrm{d}z} \mathbf{u}_1 \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} \mathrm{d}V - \frac{1}{2}\int_\Omega \mu_0 \dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}} \, \mathrm{d}V \\
&= -\int_\Omega \mu_0' \frac{\mathrm{d}c_0}{\mathrm{d}z} \mathbf{u}_1 \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} \mathrm{d}V - \int_\Omega \mu_0 \dot{\gamma}^2 \, \mathrm{d}V \, .
\end{aligned}
\tag{A.15}
$$

The same techniques used on the second term can now be directly applied to the third term, yielding

$$
\begin{aligned}
\int_\Omega \tau_Y \mathbf{u}_1 \cdot \left(\boldsymbol{\nabla} \boldsymbol{\cdot} \frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}}\right) \mathrm{d}V &= \int_{\partial\Omega} \tau_Y (\mathbf{n} \otimes \mathbf{u}_1) : \frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}} \, \mathrm{d}A - \int_\Omega \boldsymbol{\nabla}(\tau_Y \mathbf{u}_1) : \frac{\dot{\boldsymbol{\gamma}}}{\dot{\gamma}} \, \mathrm{d}V \\
&= \int_{\Lambda_B}^{\Lambda_T} \tau_Y \frac{\mathbf{u}_1}{\dot{\gamma}} \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} \mathrm{d}A - \int_\Omega \frac{\tau_Y'}{\dot{\gamma}} \frac{\mathrm{d}c_0}{\mathrm{d}z} \mathbf{u}_1 \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} \mathrm{d}V - \int_\Omega \tau_Y \dot{\gamma} \, \mathrm{d}V \, .
\end{aligned}
\tag{A.16}
$$

Observe now the fourth term on the right side of the equation cancels with two of the terms stemming from the second term and third term respectively, which becomes

$$
\int_\Omega \left(\mu_0' + \frac{\tau_Y'}{\dot{\gamma}}\right) \frac{\mathrm{d}c_0}{\mathrm{d}z} \mathbf{u}_1 \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} \mathrm{d}V = \int_\Omega \mu_0' \frac{\mathrm{d}c_0}{\mathrm{d}z} \mathbf{u}_1 \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} \mathrm{d}V + \int_\Omega \frac{\tau_Y'}{\dot{\gamma}} \frac{\mathrm{d}c_0}{\mathrm{d}z} \mathbf{u}_1 \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} \mathrm{d}V \, .
\tag{A.17}
$$

The last term is simply

$$
-\int_\Omega \mathbf{u}_1 \cdot c_1 \boldsymbol{\nabla} z \, \mathrm{d}V = -\int_\Omega c_1 w \, \mathrm{d}V \, .
\tag{A.18}
$$

Hence when we combine all of these terms, we get

$$
\begin{aligned}
\frac{1}{2} \mathrm{Re} \int_\Omega \frac{\partial \mathbf{u}_1^2}{\partial t} \mathrm{d}V = &-\int_{\Lambda_B}^{\Lambda_T} p_1 w \, \mathrm{d}A + \int_{\Lambda_B}^{\Lambda_T} \left(\mu_0 + \frac{\tau_Y}{\dot{\gamma}}\right) \mathbf{u}_1 \cdot \begin{bmatrix} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ 2\frac{\partial w}{\partial z} \end{bmatrix} \mathrm{d}A \\
&- \int_\Omega \mu_0 \dot{\gamma}^2 \, \mathrm{d}V - \int_\Omega \tau_Y \dot{\gamma} \, \mathrm{d}V - \int_\Omega c_1 w \, \mathrm{d}V
\end{aligned}
\tag{A.19}
$$

as desired.

Appendix B

Norms of Vector-valued Functions

Recall that in Section 3.1, we defined the L^p and Sobolev norms for scalar-valued functions $f\colon \Omega \to \mathbb{R}$ for some $\Omega \subseteq \mathbb{R}^n$. However, we must remind ourselves that the velocity field $\mathbf{u}_1$ is a vector-valued function $f\colon \Omega \to \mathbb{R}^3$. For the two norms in question, we must alter our definition slightly.

The L^p norm admits an easy extension. For some $\mathbf{u}\colon \Omega \to \mathbb{R}^m$, we can again define, for $p \geq 1$, the L^p norm to be

$$\|\mathbf{u}\|_p = \left(\int_\Omega |\mathbf{u}|^p \, \mathrm{d}\mathbf{x} \right)^{1/p}. \tag{B.1}$$

where $|\mathbf{u}|$ is the ℓ^2 norm of a finite-dimensional vector; that is,

$$|\mathbf{u}| = \left(\sum_{i=1}^m |u_i|^2 \right)^{1/2}. \tag{B.2}$$

It is almost immediate that all the theorems regarding scalar-valued L^p functions hold here. Since $|\mathbf{u}|$ is indeed a function from $\Omega \to \mathbb{R}$, what we have essentially constructed is attaching the L^p norm of $\mathbf{u}$ to that of $|\mathbf{u}|$.

Before we discuss the Sobolev norm, let us define $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ to be the space of linear maps (formally speaking, this is the set of $\mathbb{R}$-vector space homomorphisms) from $\mathbb{R}^n$ to $\mathbb{R}^m$, which one can think of as the space of $n \times m$ matrices, if we were to treat elements of $\mathbb{R}^n$ as column vectors. It is an elementary exercise in algebra that we have the isomorphism $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{mn}$.[7]

Consider some differentiable function $\mathbf{u}\colon \Omega \to \mathbb{R}^m$, for some $\Omega \subseteq \mathbb{R}^n$. We recall, from Definition 9.11 of [Rudin, 1976], that the (total) derivative of $\mathbf{u}$ at a point $\mathbf{x} \in \Omega$ is an element $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}) \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, so we can consider $\boldsymbol{\nabla}\mathbf{u}$ as a mapping from $\Omega \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

For a scalar-valued function $f\colon \mathbb{R} \to \mathbb{R}$, there is only one meaningful notion of a derivative, so the Sobolev norm can be easily defined as

$$\|f\|_{k,p} = \left(\sum_{i=0}^k \left\| f^{(i)} \right\|_p^p \right)^{1/p}, \tag{B.3}$$

where we sum over all the norms of the derivatives. But to extend it to higher dimensions, we have already observed that the higher total derivatives are not the same type of objects, and it seems meaningless to directly compare their norms. Hence, we must take the partial derivative instead, since $\boldsymbol{\nabla}^i \mathbf{u} = (\boldsymbol{\nabla}^i u_j) n^j \in \mathbb{R}^m$, which allows us to take the L^p norm of the same object type, getting

$$\|\mathbf{u}\|_{k,p} = \left(\sum_{|\alpha| \leq k} \|\boldsymbol{\nabla}^\alpha \mathbf{u}\|_p^p \right)^{1/p}, \tag{B.4}$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a multi-index of coordinates $\alpha_i = x_j$, and that the norm inside the sum is the L^p norm for functions from $\Omega \to \mathbb{R}^m$.

[7] In algebra, this fact is often proven in a more general setting for free modules over a commutative ring. See perhaps Section 10.3 of [Dummit and Foote, 2003].